Class Work 1

This is to be completed together as a class, to establish shared notation and terminology.

The dihedral group D_n is the group of symmetries of a regular n-gon. It consists of n-reflections and n-rotations. The group C_n is the group of rotations of the n-gon. It is a subgroup of D_n.

A set of elements $S \subset G$ of a group G **generates** G if every element of G is the product of elements of S and the inverses of the elements of S (allowing repetitions.). A nontrivial finite group is **cyclic** if it is has a generating set with a single element.

The **order** of a group is its cardinality. The **order** of a group element g is the smallest non-negative integer n such that $g^n = \mathbb{1}$.

1. Generators

(1) Draw a regular octagon and label its reflections and rotations.

(2) What rotations of the octagon generate C_8?

(3) If S generates D_8, is it possible that S contains only rotations?

(4) Find a single rotation and a single reflection that generate D_8.

(5) Does *every* pair of rotation and reflection generate D_8? If not, what are the restrictions?

(6) Find two reflections that generate D_8 and two reflections that do not generate D_8.

(7) How does the previous discussion change if 8 is changed to some other number?

2. Subgroups

If H is a subgroup of G, we write $H < G$. For $g \in G$, the **left coset** of g is

$$gH = \{gh | h \in H\}.$$

The set of left cosets is denoted G/H and is a partition of G (whether G is finite or infinite). The cardinality of G/H is denoted $[G : H]$ and is called the **index** of H in G.

(1) The group C_8 is a cyclic subgroup of D_8. What are the other cyclic subgroups?

(2) Show that D_4 is a subgroup of D_8. Are there distinct subgroups of D_8, each isomorphic to D_4?

(3) Explain why if $H < G$ and $h \in H$ then $hH = H$.

(4) Do we remember Lagrange's Theorem for finite groups? Do we remember how to prove it?

(5) Do this several times: Pick a subgroup $H < D_8$ and $g \in D_8$. Find the elements of the coset gH.

(6) Explain why all subgroups of a cyclic group are cyclic.

3. The orbit-stabilizer theorem

Suppose that G is a subgroup of a group of symmetries of some object X. Recall that, among other things, this means that each element $g \in G$ is a function $g\colon X \to X$. Let $x \in X$. The **orbit** of x under G is the set of points in X that are the image of x under some symmetry in G.

$$\textsc{orb}_G(x) = \{y \in X : \exists g \in G \text{ s.t. } g(x) = y\}$$

The **stabilizer** of x under G is:

$$\textsc{stab}_G(x) = \{g \in G : g(x) = x\}.$$

We sometimes omit the G if it is clear from context.

(1) Summarize the big difference between $\textsc{orb}(x)$ and $\textsc{stab}(x)$.

(2) For different choices of n and different choices of point x in a regular n-gon (vertex, midpoint of an edge, non-midpoint of an edge), calculate $\textsc{orb}_G(x)$ and $\textsc{stab}_G(x)$ where $G = D_n$.

(3) Show that there is a bijection between $G/\textsc{stab}_G(x)$ and $\textsc{orb}_G(x)$.

(4) Prove:

Theorem (Orbit-Stabilizer theorem)**.** *If G is a subgroup of the group of symmetries of an object X and if $x \in X$, then*

$$|G| = |\textsc{stab}_G(x)||\textsc{orb}_G(x)|.$$

(5) Use the orbit-stabilizer theorem to compute the order of the group of orientation-preserving symmetries of the dodecahedron.

(6) Suppose that G is a subgroup of a group of symmetries of an object X. Let $\mathcal{O} = \{\textsc{orb}_G(x) : x \in X\}$. Prove that $\mathcal{O}$ is a partition of X. In particular, if X is finite, then

$$|X| = \sum_{O \in \mathcal{O}} |O|$$

(7) Recall that a permutation of a set X is just another word for a bijection $X \to X$. The permutations of any set X form a group. Ignoring all structure on X, we think of $\textsc{perm}(X)$ as the set of symmetries of X, thought of as just a set of points.

Prove:

Theorem (Cayley's Theorem)**.** *For every group G, there is a set X such that G is (isomorphic to) a subgroup of $\textsc{perm}(X)$.*

(Hint: Set $X = G$. For each $g \in G$, let $m_g\colon X \to X$ be defined by $m_g(x) = gx$ (i.e. multiply g and x). Consider the map $g \mapsto m_g$.)

Definition 3.1. An **action** of a group G on an object X is a homomorphism $\phi\colon G \to \textsc{sym}(X)$. It is **faithful** if it is injective (i.e. a monomorphism). The homomorphism is also called a **representation**. If G acts on X, we often write $G \curvearrowright X$. For $g \in G$ and $x \in X$, we often let $g \cdot x = \phi(g)(x)$, in which case it is a **left action** of G on X.

Cayley's theorem says that every group has a faithful action on itself. The Sylow theorems are important classical theorems concerning subgroups of a finite group of prime power. They are proved by studying orbits and stabilizers of elements of G under the left action of G on itself.