Thm If G acts freely on $\mathbb{E}^n$ then G is torsion-free.

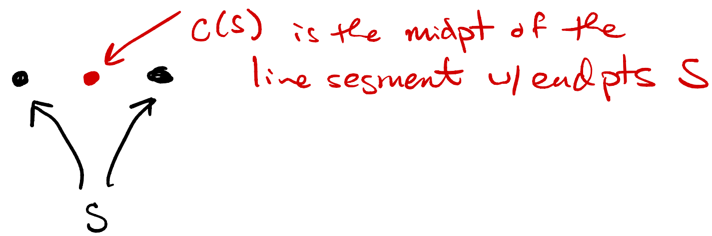
Before proving this we establish some geometric facts.

Recall $\mathbb{E}^n$ is $\mathbb{R}^n$ with the Eucl. distance metric $d(x,y) = \|x-y\|$

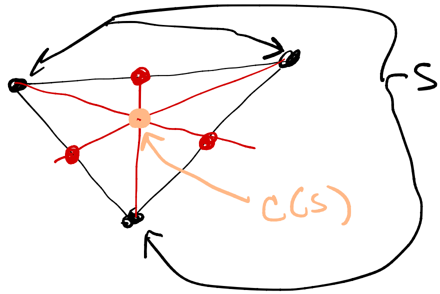
Fact we won't prove: If $T : \mathbb{E}^n \to \mathbb{E}^n$ is an isometry then for every line segment $L \subset \mathbb{E}^n$ $T(L)$ is also a line segment.

Def. Suppose $S \subset \mathbb{E}^n$ is a finite set the **centroid** is the point $\frac{1}{|S|} \sum_{x \in S} x = c(S)$

Example ①



②



Lemma Suppose $S \subset \mathbb{E}^n$ and $2 \leq |S| < \infty$. Let $x \in S$. Then $c(S)$ lies on the line segment with endpts $c(S \setminus x)$ and x. In fact,

$$c(S) = \left(1 - \frac{|S|-1}{|S|}\right) x + \frac{|S|-1}{|S|} c(S \setminus x)$$

proof

$$\left(1 - \frac{|S|-1}{|S|}\right) x + \frac{|S|-1}{|S|} c(S \setminus x) =$$

$$\frac{1}{|S|} x + \frac{|S|-1}{|S|} \underline{c(S \setminus x)} =$$

definition

$$\frac{1}{|S|} \left(x + \frac{|S|-1}{1} \cdot \overbrace{\frac{1}{|S|-1} \sum_{y \in S \setminus x} y}\right) =$$

$$\frac{1}{|S|} \left(x + \sum_{y \in S \setminus x} y \right) = \frac{1}{|S|} \sum_{y \in S} y = c(S).$$

$\square$

Now we prove:

Thm If G acts freely on $\mathbb{E}^n$ then G is torsion-free.

pf We prove the contrapositive. Suppose $g \in G$ has finite order. Let $H < G$ be the subgroup generated by g. that is

$$H = \{ \mathbb{1}, g, g^2, \dots, g^{\operatorname{ord}(g)-1} \}$$

Choose $x \in \mathbb{E}^n$ and let $S = \operatorname{orb}_H(x)$ so $|S| = \operatorname{order}(g)$. Let $c(S)$ be the centroid of S. Notice that $\forall\, y \in S$, $\forall\, h \in H$, $y = g^k \cdot x$ $h = g^m$ for some k, m so $h \cdot y = g^m g^k \cdot x = g^{m+k} \cdot x$ so $h \cdot y \in S$. Since $h \in G$ is an isometry, the function $\phi : S \to S$ defined by $\phi(s) = h \cdot s$ is injective. Since S is finite, ϕ is a bijection. Thus, $h \cdot S = S$ for every $h \in H$. Thus by our previous work:

$$g \cdot c(S) \underset{\substack{\uparrow \\ \text{Lemma}}}{=} c(g \cdot S) \underset{\substack{\uparrow \\ g \cdot S = S}}{=} c(S) \quad \text{so } g \in \operatorname{stab}(c(S))$$

Thus either $g = \mathbb{1}$ or the action of G on $\mathbb{E}^n$ is not free. $\square$

lemma Let $T : \mathbb{E}^n \to \mathbb{E}^n$ be an isometry.

If $S \subset \mathbb{E}^n$ is finite, then

$$T\big(c(S)\big) = c\big(T(S)\big)$$

("isometries move centroids to centroids")

pf Number the elements of S as

$$x_1, x_2, \dots, x_n$$

Let L_i be the line segment from x_i to $c(\{x_1, \dots, x_{i-1}\})$ for $2 \leq i \leq n$

T takes L_i to the line segment joining $T(x_i)$ and $c(\{T(x_1), \dots, T(x_{i-1})\})$

As $c(S)$ lies $\frac{n-1}{n}$ of the way along the line segment L_n, the result follows by induction on $|S|$.