Selected Course Notes for MA 262: Vector Calculus

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Contents

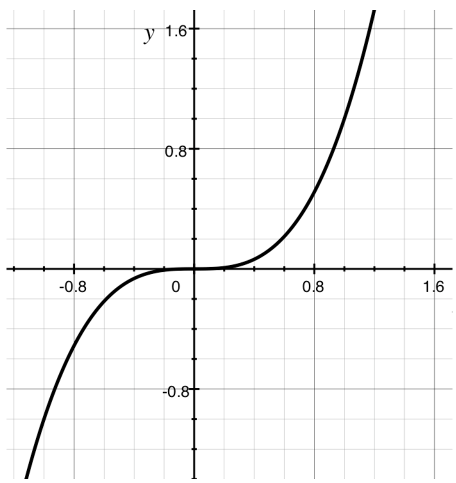
1. Functions and their derivatives

This section takes a look at some functions you may have encountered and interprets them in various ways. Each of these ways will be studied and generalized in this course.

1.1. **A function $f\colon \mathbb{R} \to \mathbb{R}$.** A function $f\colon \mathbb{R} \to \mathbb{R}$ takes in a real number and spits out a real number. For example, $f(x) = x^3$. Here are five ways of thinking about such a function and its derivative.

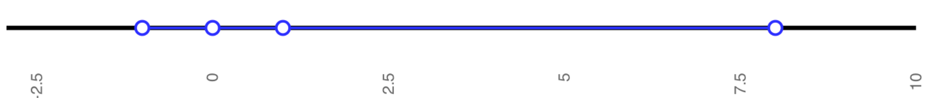
Viewpoint 1: The Graph

The graph of $f(x)$ is the set of points $(x, y) \in \mathbb{R}^2$ such that $y = f(x)$. The derivative $f'(a)$ is the slope of the line tangent to the graph at the point $(a, f(a))$



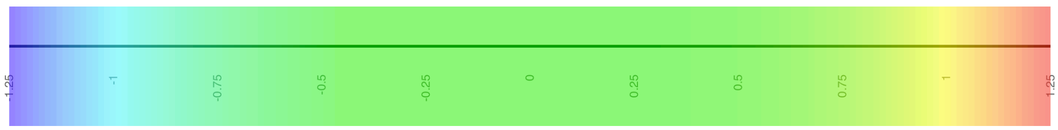
Viewpoint 2: t is time and $f(t)$ is position

In the figure, the labels below the line correspond to position $f(t)$. Blue dots corresponding to times $t = -1, 0, 1, 2$ have been placed on the line. The derivative $f'(t)$ is the velocity (speed and direction) of the object at time t.



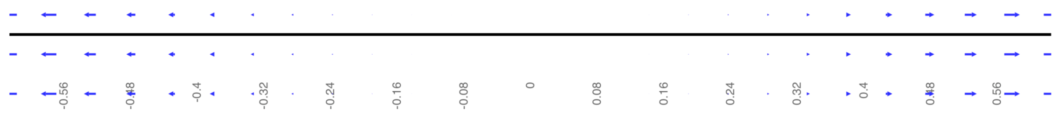
Viewpoint 3: x represents position and $f(x)$ represents temperature (or amount)

In the image, the labels below the line correspond to x and the colors correspond to $f(x)$ with warmer colors representing a greater temperature (or amount) and cooler colors representing smaller temperatures (or amounts.) This is called a **scalar field**. The derivative $f'(x)$ represents the direction and amount of greatest increase in temperature.



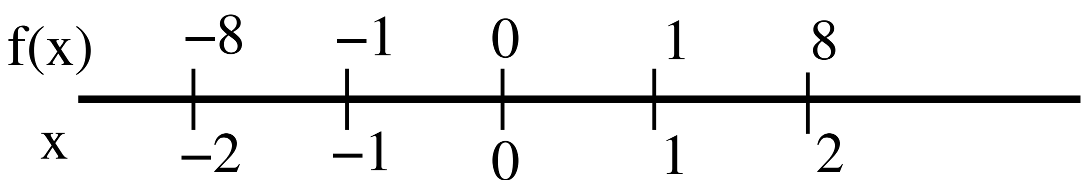
Viewpoint 4: x is position and $f(x)$ is a direction.

In the image, the labels below the line correspond to x and the arrows correspond to $f(x)$. If $f(x) < 0$, the arrow points left; if $f(x) > 0$, the arrow points right. The length of the arrow corresponds to $|f(x)|$. The derivative $f'(x)$ seems to measure how much stuff is flowing into or out of x.



Viewpoint 5: x is position and $f(x)$ is position.

In this case f is a change-of-coordinates function. Each point on the line has two labels, we get between them by using f (or its inverse $f^{-1}(x) = \sqrt[3]{x}$.



The absolute value of the derivative measures how much (infinitessimal) stretching occurs at a point. Consider, for example, the interval $[2, 2+h]$. When we plug the points from this interval into the function f, we get another interval $[f(2), f(2+h)]$. The ratio of the lengths of these intervals is:

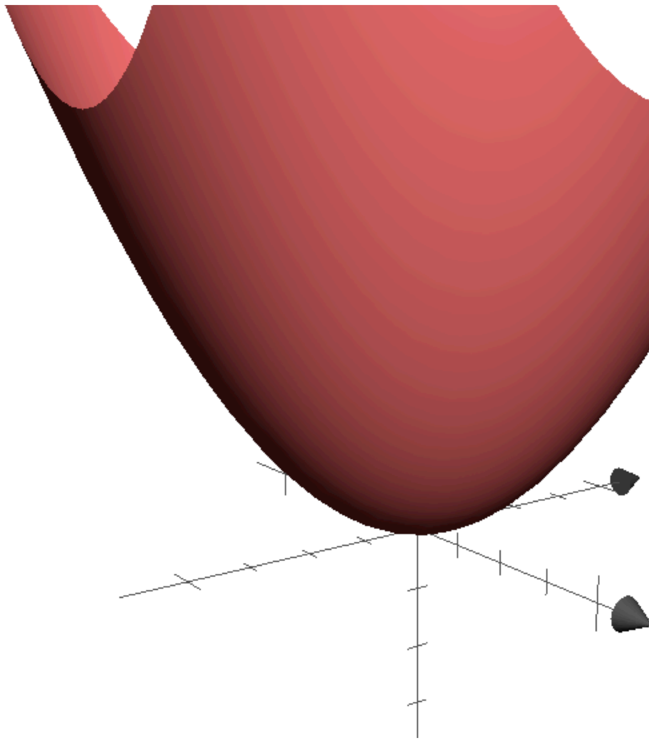
$$\frac{f(2+h) - f(2)}{(2+h) - 2} = \frac{f(2+h) - f(2)}{h}.$$

This ratio measures how much the interval $[2, 2+h]$ is stretched by the function f. If we take the limit as h goes to zero, we get a measurement of the stretching that occurs at the point $x = 2$. But this limit is exactly the derivative $f'(2)$.

1.2. **A function** $f\colon \mathbb{R}^2 \to \mathbb{R}$. Such a function takes in a vector (x, y) in $\mathbb{R}^2$ and spits out a number $f(x, y)$ in $\mathbb{R}$. For example, $f(x, y) = x^2 + y^2$.

Viewpoint 1: The Graph

The graph is the set of points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$.



The gradient

$$\nabla f(x, y) = \begin{pmatrix} \frac{\partial}{\partial x} f(x, y) \\ \frac{\partial}{\partial y} f(x, y) \end{pmatrix}$$

is the vector pointing in the direction of greatest increase in z at $(x, y) \in \mathbb{R}^2$. The tangent plane to the graph of f at the point $\mathbf{a}$ has equation:

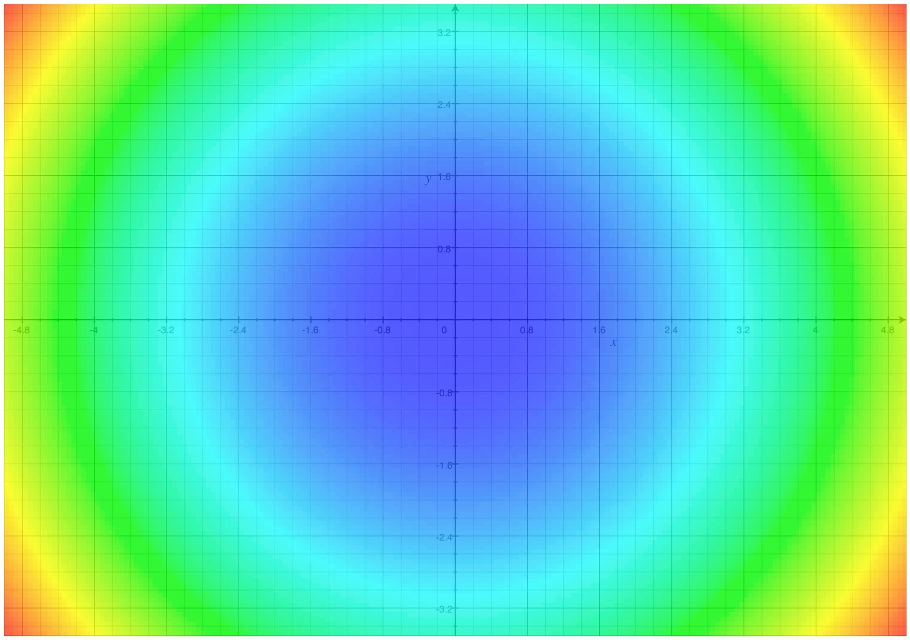
$$L(\mathbf{x}) = \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + f(\mathbf{a}).$$

This is also the equation of the best linear (or affine) approximation to f near $\mathbf{a}$.

Viewpoint 2: Is not relevant, since the input to the function is two-dimensional and so we shouldn't think of it as being time.

Viewpoint 3: $\mathbf{x}$ is position and $f(\mathbf{x})$ is temperature (or amount).

As with functions from $\mathbb{R}$ to $\mathbb{R}$, this viewpoint makes sense for functions from $\mathbb{R}^2$ to $\mathbb{R}$. Below is a temperature plot of this scalar field:



The gradient $\nabla f(x, y)$ points in the direction of greatest temperature increase.

Viewpoint 4: Does not make sense since the 1-dimensional output cannot express a direction in 2-dimensions.

Viewpoint 5: Does not make sense since the input and output to f have different dimensions.

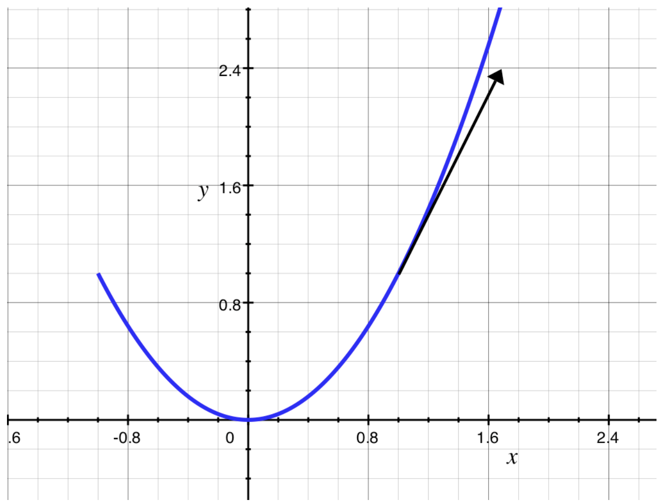
1.3. **A function** $f\colon \mathbb{R} \to \mathbb{R}^2$**.** Such a function takes in a real number and spits out a 2-dimensional vector. For example, $f(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$.

Viewpoint 1: The Graph

Technically this makes sense. We could plot the points $(x, y, z) \in \mathbb{R}^3$ such that $(y, z) = f(x)$. But we usually don't do this.

Viewpoint 2: t represents time, and $f(t)$ represents position.

This makes sense. For each time t, we can plot the point (t, t^2). We obtain a curve in $\mathbb{R}^2$. We can label this curve with the direction we are travelling on it and label particular points with what time we are at that point.



The derivative $f'(t) = (1, 2t)$ tells us what direction we are travelling in at each t value and $||f'(t)|| = \sqrt{1 + 4t^2}$ tells us our speed.

Viewpoints 3 - 5: These do not make sense for this function. Why is that?

1.4. **A function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$.** Such a function takes in a vector in $\mathbb{R}^2$ and spits out another vector in $\mathbb{R}^2$. For example,

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - y \\ x + y \end{pmatrix}.$$

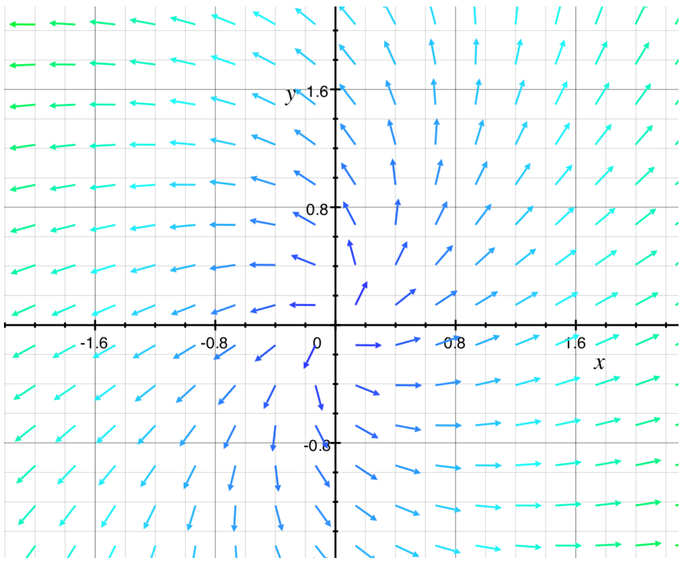
Viewpoint 1: The Graph

Technically, this makes sense. We could plot points $(x, y, z, w) \in \mathbb{R}^4$ such that $(z, w) = f(x, y)$. But we usually don't do this.

Viewpoints 2 - 3: These don't make sense in this context. Why is that?

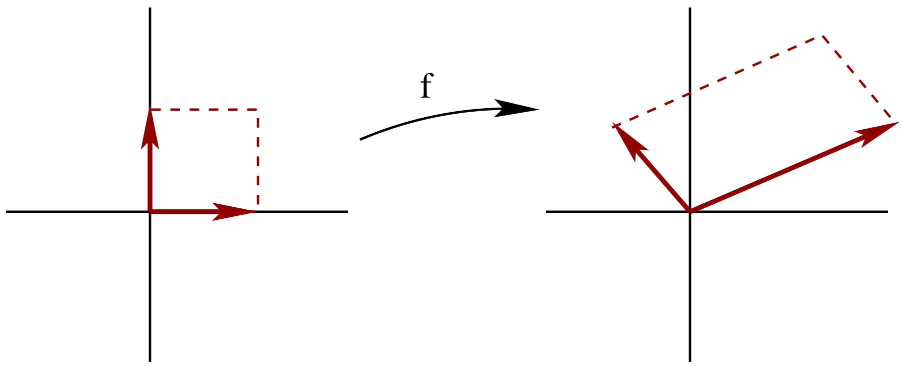
Viewpoint 4: $\mathbf{x}$ is position and $f(\mathbf{x})$ is position.

This is called a vector field. At each point $(x, y) \in \mathbb{R}^2$ we draw an arrow of length $||f(\mathbf{x})||$ and pointing in the direction of $f(\mathbf{x})$. In the computer image below, the color of the arrow indicates how long it is.



Viewpoint 5: $\mathbf{x}$ and $f(\mathbf{x})$ both represent position.

We say that f is a change of coordinates function. Under this viewpoint, each point of the plane is labelled with both $\mathbf{x}$ and with $f(\mathbf{x})$. We think of f as a transformation taking one coordinate system to the other.



Notice that the unit vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ define a square R of area 1. After applying f, this square is transformed into a parallelogram defined by the vectors $(2, 1)$ and $(-1, 1)$. The area of the parallelogram $f(R)$ is equal to $\left| \det \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \right| = 3$. So the transformation f scales the area of R by 3. It turns out that this is related to the derivative of f.

2. Derivatives and Differentiability

In Calc II, you learn that given a (differentiable) function

$$f\colon \mathbb{R}^n \to \mathbb{R}$$

you can find its gradient:

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

For reasons that will be explained later, we define the **derivative** of f to be the gradient, but turned into a row matrix:

$$Df = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \dots & \frac{\partial f}{\partial x_n} \end{pmatrix}$$

Notice that $\nabla f(\mathbf{x})$ is a *vector* but $Df(\mathbf{x})$ is not.

Example 2.1. Let $f(x, y) = x^2y - y^3$ Then:

$$\nabla f(\mathbf{x}) = \begin{pmatrix} 2xy \\ x^2 - 3y \end{pmatrix}$$

and

$$Df(\mathbf{x}) = \begin{pmatrix} 2xy & x^2 - 3y \end{pmatrix}$$

Suppose now that we have a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$. The output of f is a vector in $\mathbb{R}^n$ and so has n-coordinates – each of which depends on m-independent variables. That is:

$$f(x_1, \dots, x_n) = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ f_2(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{pmatrix}.$$

Each function f_i has a derivative (and a gradient). We define the **derivative** of f at $\mathbf{a} \in \mathbb{R}^n$ to be simply the matrix whose ith row is the derivative of f_i. That is:

$$Df(\mathbf{a}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \frac{\partial f_1}{\partial x_2}(\mathbf{a}) & \frac{\partial f_1}{\partial x_3}(\mathbf{a}) & \dots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{a}) & \frac{\partial f_2}{\partial x_2}(\mathbf{a}) & \frac{\partial f_2}{\partial x_3}(\mathbf{a}) & \dots & \frac{\partial f_2}{\partial x_n}(\mathbf{a}) \\ \frac{\partial f_3}{\partial x_1}(\mathbf{a}) & \frac{\partial f_3}{\partial x_2}(\mathbf{a}) & \frac{\partial f_3}{\partial x_3}(\mathbf{a}) & \dots & \frac{\partial f_3}{\partial x_n}(\mathbf{a}) \\ \vdots & & & \vdots & \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \frac{\partial f_m}{\partial x_2}(\mathbf{a}) & \frac{\partial f_m}{\partial x_3}(\mathbf{a}) & \dots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{pmatrix}$$

The entry in the ith row and jth column is the partial derivative at $\mathbf{a}$ of f_i with respect to x_j. Equivalently, the ith row consists of $Df_i(\mathbf{a})$.

Example 2.2. If $f(x, y) = x^2 + 3yx$ then $Df(x, y) = \begin{pmatrix} 2x + 3y & 3x \end{pmatrix}$.

Example 2.3. If $f(x,y) = \begin{pmatrix} 2x - 3y \\ x + y \end{pmatrix}$ then $Df(x,y) = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$. Notice that this is the transpose of the gradient fo f.

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3. Some Linear Algebra

3.1. Matrix Multiplication. As you know we can write a vector $\mathbf{a} \in \mathbb{R}^n$ in either horizontal

$$(a_1, a_2, \ldots, a_n)$$

or vertical

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

format.

A row vector is different. It is written horizontally without commas:

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_n \end{pmatrix}.$$

If we have a row vector $\mathbf{a} = \begin{pmatrix} a_1 & \ldots & a_n \end{pmatrix}$ and a column vector $\mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$, we define their product:

$$\mathbf{ab} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n.$$

This should remind you of the dot product. In fact if we let $\mathbf{a}^T$ denote the column vector obtained by writing $\mathbf{a}$ as a column instead of as a row,

$$\mathbf{ab} = \mathbf{a}^T \cdot \mathbf{b}.$$

3.1.1. *Matrix times Vector.* We explain how to take a matrix times a vector. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{pmatrix}$$

and let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

We define

$$A\mathbf{x} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{pmatrix}.$$

Notice that each entry in the new vector is the product of the corresponding row of A with $\mathbf{x}$. So if we let $\mathbf{a}_i$ be the ith row of A, for $1 \leq i \leq m$ we can write the multiplication as:

$$A\mathbf{x} = \begin{pmatrix} \mathbf{a}_1\mathbf{x} \\ \mathbf{a}_2\mathbf{x} \\ \vdots \\ \mathbf{a}_m\mathbf{x} \end{pmatrix}.$$

Example 3.1. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$. Then:

$$A\mathbf{x} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -3 \\ 7 \end{pmatrix} = \begin{pmatrix} 11 \\ 19 \end{pmatrix}$$

Example 3.2. Let $A = \begin{pmatrix} 0 & -2 & 1 \\ 3 & 0 & -4 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$. Then:

$$A\mathbf{x} = \begin{pmatrix} 0 & -2 & 1 \\ 3 & 0 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ 11 \end{pmatrix}$$

Suppose that A is an $m \times n$ matrix and that $\mathbf{b} \in \mathbb{R}^m$. We can define a function $L\colon \mathbb{R}^n \to \mathbb{R}^m$ by

$$L(\mathbf{x}) = A\mathbf{x} + \mathbf{b}.$$

Such a function is called a linear or affine function.

(Remark: Technically, a linear function should take the zero vector to the zero vector. A function which is a linear function plus a constant function should be called an affine function. However, it is traditional in calculus to blur the distinction.)

3.1.2. *Matrix times Matrix.* While we're at it we can now define the product of two matrices, assuming that the dimensions work out okay.

Suppose that A is an $m \times n$ matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

Let $\mathbf{a}_1$ denote the first row, $\mathbf{a}_2$ the second row, etc. Write:

$$A = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{pmatrix}$$

Suppose that B is an $n \times p$ matrix:

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1p} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2p} \\ \vdots \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{np} \end{pmatrix}$$

Let $\mathbf{b}_1$ denote the first column, $\mathbf{b}_2$ the second column, etc. Write

$$B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \dots & \mathbf{b}_p \end{pmatrix}.$$

Define the product of A and B to be

$$AB = \begin{pmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2 & \dots & \mathbf{a}_1\mathbf{b}_p \\ \mathbf{a}_2\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 & \dots & \mathbf{a}_2\mathbf{b}_p \\ \vdots \\ \mathbf{a}_m\mathbf{b}_1 & \mathbf{a}_m\mathbf{b}_2 & \dots & \mathbf{a}_m\mathbf{b}_p \end{pmatrix}$$

Notice this is the $m \times p$ matrix whose entry in the ith row and jth column is the product of the ith row of A with the jth column of B.

Example 3.3. Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$. Then

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - y \\ x + y \end{pmatrix}.$$

Example 3.4. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$. Let $B = \begin{pmatrix} 5 & 7 \\ 0 & 6 \end{pmatrix}$. Then

$$AB = \begin{pmatrix} 2(5) + 3(0) & 2(7) + 3(6) \\ 1(5) + (-1)(0) & 1(7) + (-1)(6) \end{pmatrix} = \begin{pmatrix} 10 & 32 \\ 5 & 1 \end{pmatrix}.$$

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4. Differentiability and the uses of the derivative

4.1. **Differentiability.** We are now ready to define what it means to be differentiable. The concept is modelled on the definition from MA 122, so you should review those definitions.

Definition: A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is **differentiable** at $\mathbf{a} \in \mathbb{R}^n$ if there exists an affine function $L\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

- $f(\mathbf{a}) = L(\mathbf{a})$
- $$\lim_{\mathbf{x}\to\mathbf{a}} \frac{||L(\mathbf{x}) - f(\mathbf{x})||}{||\mathbf{x}-\mathbf{a}||} = 0$$

A few remarks:

(1) The affine function L is called the "affine approximation" to f at $\mathbf{a}$. The criteria ensure that the relative error between L and f goes to zero and that they are actually equal at $\mathbf{a}$.
(2) In past math classes, L was probably called the "linear approximation" to f, but it is, in fact, an affine function that is not necessarily linear.
(3) Since L is an affine function, there exists an $m \times n$ matrix A such that $L(\mathbf{x}) = A(\mathbf{x}-\mathbf{a}) + f(\mathbf{a})$. This matrix is called the **derivative** of f at $\mathbf{a}$ and is denoted $Df(\mathbf{a})$.
(4) If f is differentiable and if $Df(\mathbf{x})$ is continuous, then we say that f is C^1.

4.2. **Differentiability.** Throughout this section, let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function and let f_i be the ith coordinate function. That is $f(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_m(\mathbf{x}))$.

Theorem 4.1. Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}^m$ has the property that each component function f_i is differentiable at $\mathbf{a}$. Then f is differentiable at $\mathbf{a}$. Furthermore, $f_i\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a}$, if there is an open ball X containing $\mathbf{a}$ such that f_i is defined on X and all partial derivatives of f_i exist and are continuous on X.

Example 4.2. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$f(x,y,z) = (\ln(|xyz|), x+y+z^2)$$

Then

$$Df(x,y,z) = \begin{pmatrix} 1/x & 1/y & 1/z \\ 1 & 1 & 2z \end{pmatrix}.$$

Let A be the coordinate axes in $\mathbb{R}^3$. That is, $A = \{(x,y,z) : xyz = 0\}$. Each entry in the matrix $Df(x,y,z)$ is continuous on $\mathbb{R}^3 - A$. The function f is defined on $\mathbb{R}^3 - A$. Consequently, f is differentiable at each point $\mathbf{a} \in \mathbb{R}^3 - A$.

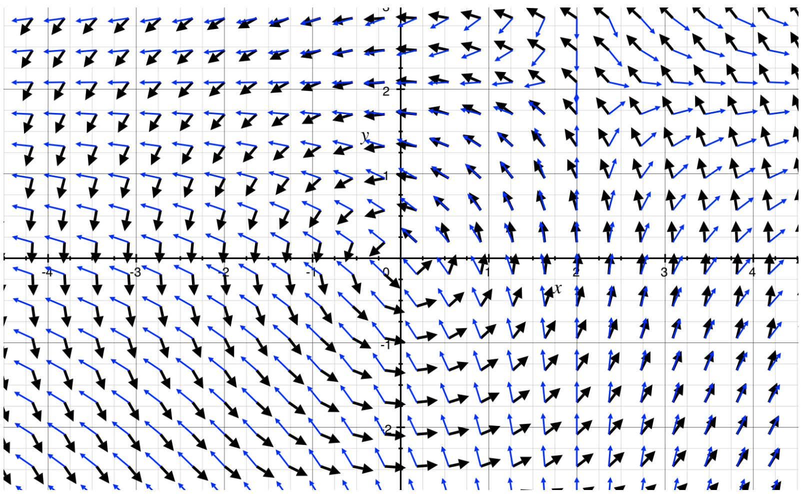
4.3. **Uses of the Derivative.**

4.3.1. *Affine Approximation.*

Example 4.3. Let $F(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix}-y\\ x\end{pmatrix}$. The affine approximation to F at $(1,1)$ is

$$L(x,y) = \frac{1}{4}\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}x-1\\ y-1\end{pmatrix} + \begin{pmatrix}-1/2\\ 1/2\end{pmatrix}.$$

The figure below shows the vector field F in black and the linear approximation L in blue. Notice that near the point $(1,1)$ the black and blue arrows (nearly) coincide but far away they can be radically different.



Affine approximations are traditionally called "linear approximations" because if we base our coordinate system of the domain at (a,b) and of the range at $f(a,b)$, the affine approximation is actually a linear function. We will work more with these coordinate systems later.

4.3.2. *Volume Approximation.* It is a fact that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 change of coordinates function then the amount that f scales n-dimensional volume near $\mathbf{a}$ is approximately the absolute value of the determinant of $Df(\mathbf{a})$. Here is an example:

Example 4.4. Define $f(r,\theta) = \begin{pmatrix}r\cos\theta\\ r\sin\theta\end{pmatrix}$. The derivative of f is

$$Df(r,\theta) = \begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}.$$

This has determinant equal to $r\cos^2\theta + r\sin^2\theta = r$. In the $r-\theta$ plane, the square $[r, r+\Delta r] \times [\theta, \theta+\Delta\theta]$ has area $\Delta\theta\Delta r$. Applying f to this square converts it into the set of (x,y) values between the circle of radius r and the circle of radius $r+\Delta r$ and with angle between θ and $\theta+\Delta\theta$. Elementary geometry guarantees that the area of this region is

$$(\frac{\Delta\theta}{2\pi})\pi(r+\Delta r)^2 - (\frac{\Delta\theta}{2\pi})\pi r^2 = (\Delta\theta/2)(2r\Delta r + (\Delta r)^2).$$

The actual scaling factor is, therefore,

$$\frac{1}{2}(2r + \Delta r)$$

which goes to r as $\Delta r \to 0^+$.

Example 4.5. Define $f\colon \mathbb{R}^2 \to \mathbb{R}^4$ by

$$f(x, y) = (xy, x^2y, xy^3, x^4e^y)$$

Then

$$Df(x, y) = \begin{pmatrix} y & x \\ 2xy & x^2 \\ y^3 & 3xy^2 \\ 4x^3e^y & x^4e^y \end{pmatrix}$$

and

$$Df(1, 2) = \begin{pmatrix} 2 & 1 \\ 4 & 1 \\ 8 & 12 \\ 4e^2 & e^2 \end{pmatrix}.$$

Here is another example, demonstrating an important point (to be made later).

Example 4.6. Define $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{aligned} f(x, y) &= (x^2 + 2x, e^y) \\ g(x, y) &= (\sin(x), 5y + x) \end{aligned}$$

Notice that we can compose f and g to obtain $f \circ g\colon \mathbb{R}^2 \to \mathbb{R}$. A formula for $f \circ g$ is:

$$f \circ g(x, y) = (\sin^2 x + 2\sin x, e^{5y+x}).$$

Notice that $g(0, 0) = (0, 0)$.

Compare $Df(g(\mathbf{0}))$, $Dg(\mathbf{0})$ and $D(f \circ g)(\mathbf{0})$.

Solution:

$$\begin{aligned} Df(\mathbf{0}) &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \\ Dg(\mathbf{0}) &= \begin{pmatrix} 1 & 0 \\ 1 & 5 \end{pmatrix} \\ D(f \circ g)(\mathbf{0}) &= \begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix} \end{aligned}$$

Notice that:

$$D(f \circ g)(\mathbf{0}) = Df(g(\mathbf{0}))Dg(\mathbf{0}).$$

This is an example of the chain rule at work.

4.4. **The chain rule.** Here is the justly famous chain rule:

Theorem 4.7. Suppose that $g\colon \mathbb{R}^n \to \mathbb{R}^m$ and $f\colon \mathbb{R}^m \to \mathbb{R}^p$ are functions which are defined on open sets $Y \subset \mathbb{R}^n$ and $X \subset \mathbb{R}^m$ such that $g(Y) \subset X$. Assume that g is differentiable at $\mathbf{y} \in Y$ and that f is differentiable at $g(\mathbf{y}) \in X$. Then, $f \circ g\colon \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at $\mathbf{y}$ and $D(f \circ g)(\mathbf{y}) = Df\big(g(\mathbf{y})\big)Dg(\mathbf{y})$.

Example 4.8. Define $f(x, y) = (x^2, x^2 + y^2)$. Let $\hat{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the function f with domain in polar coordinates. What is $D\hat{f}(r, \theta)$?

Solution: Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the change from polar coordinates to rectangular coordinates. That is,

$$T(r, \theta) = (r\cos\theta, r\sin\theta).$$

Then, by definition, $\hat{f} = f \circ T$. Since the coordinates of f and T are polynomials and trig functions, f and T are everywhere differentiable. A calculation shows that:

$$Df(x, y) = \begin{pmatrix} 2x & 0 \\ 2x & 2y \end{pmatrix}.$$

Thus,

$$Df(T(r, \theta)) = \begin{pmatrix} 2r\cos\theta & 0 \\ 2r\cos\theta & 2r\sin\theta \end{pmatrix}.$$

Another calculation shows that

$$DT(r, \theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

Thus, by the chain rule:

$$D\hat{f}(r, \theta) = \begin{pmatrix} 2r\cos\theta & 0 \\ 2r\cos\theta & 2r\sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = \begin{pmatrix} 2r\cos^2\theta & -2r^2\cos\theta\sin\theta \\ 2r & 0 \end{pmatrix}$$

The point is that "the derivative of a composition is the product of derivatives".

Sketch of proof of Chain Rule. Let $g\colon \mathbb{R}^n \to \mathbb{R}^m$ and $f\colon \mathbb{R}^m \to \mathbb{R}^k$ be such that g and f are both differentiable at $\mathbf{0}$ and $g(\mathbf{0}) = \mathbf{0}$ and $f(\mathbf{0}) = \mathbf{0}$.

Special case: f and g are both linear.

Then there exist matrices A_{mk} and B_{nm} so that

$$\begin{aligned} f(\mathbf{x}) &= A\mathbf{x} \qquad \text{for all } \mathbf{x} \in \mathbb{R}^m \\ g(\mathbf{x}) &= B\mathbf{x} \qquad \text{for all } \mathbf{x} \in \mathbb{R}^n \end{aligned}$$

This implies that, for all $\mathbf{x} \in \mathbb{R}^n$

$$f \circ g(\mathbf{x}) = A(B\mathbf{x}) = (AB)\mathbf{x}.$$

Notice that:

$$\begin{aligned} Df(g(\mathbf{0})) &= A \\ Dg(\mathbf{0}) &= B \\ D(f \circ g)(\mathbf{0}) &= AB \end{aligned}$$

Thus,

$$D(f \circ g)(\mathbf{0}) = Df(g\mathbf{0})Dg(\mathbf{0})$$

as desired.

General Case: f and g are not necessarily linear.

Since $g\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{0}$, for $\mathbf{x}$ near $\mathbf{0}$,

$$g(\mathbf{x}) \approx Dg(\mathbf{0})\mathbf{x}.$$

Similarly, since $f\colon \mathbb{R}^m \to \mathbb{R}^k$ is differentiable at $g(\mathbf{0}) = \mathbf{0}$, for $\mathbf{x}$ near $\mathbf{0}$,

$$f(\mathbf{x}) \approx Df(g(\mathbf{0}))\mathbf{x}.$$

To prove the theorem we just need to show that

$$f \circ g(\mathbf{x}) \approx Df(g(\mathbf{0}))Dg(\mathbf{0}).$$

Remember that $\approx$ in this context means that the relative error goes to 0 as $\mathbf{x} \to \mathbf{0}$. We didn't go over this in class, but here is a proof:

For convenience, define the following:

$$\begin{aligned} B &= Dg(\mathbf{0}) \\ A &= Df(\mathbf{0}) \end{aligned}$$

We need to show that for each $\epsilon > 0$, there exists $\delta > 0$ so that if $0 < ||\mathbf{x}|| = ||\mathbf{x} - \mathbf{0}|| < \delta$ then

$$\frac{||f \circ g(\mathbf{x}) - AB\mathbf{x}||}{||\mathbf{x} - \mathbf{0}||} < \epsilon.$$

Notice that:

$$||f \circ g(\mathbf{x}) - AB\mathbf{x}|| = ||f \circ g(\mathbf{x}) - Ag(\mathbf{x}) + Ag(\mathbf{x}) - AB\mathbf{x}||.$$

By the triangle inequality,

$$||f \circ g(\mathbf{x}) - AB\mathbf{x}|| \leq ||f \circ g(\mathbf{x}) - Ag(\mathbf{x})|| + ||A(g\mathbf{x}) - B\mathbf{x})||.$$

Now there exists a constant α, such that for all $\mathbf{y} \in \mathbb{R}^m$, $||A\mathbf{y}|| \leq \alpha||\mathbf{y}||$. Thus,

$$\begin{aligned} ||f \circ g(\mathbf{x}) - AB\mathbf{x}|| &\leq \\ ||f(g(\mathbf{x})) - Ag(\mathbf{x})|| + ||A(g\mathbf{x}) - B\mathbf{x})|| &\leq \\ ||f(g(\mathbf{x})) - Ag(\mathbf{x})|| + \alpha||g(\mathbf{x} - B\mathbf{x}|| & \end{aligned}$$

We now consider the relative errors.

Piece 1: Since g is differentiable at $\mathbf{0}$, there exists $\delta_1 > 0$, so that if $0 < ||\mathbf{x}|| < \delta_1$ then

$$\frac{||g(\mathbf{x}) - B\mathbf{x}||}{||\mathbf{x}||} < \epsilon/2\alpha.$$

Piece 2: There is a theorem, which guarantees that (since g is differentiable at $\mathbf{0}$) there exists $\delta_2 > 0$ so that if $||\mathbf{x}|| < \delta_2$, then there is a constant β such that

$$||g(\mathbf{x})|| \leq \beta||\mathbf{x}||.$$

Piece 3: Since f is differentiable at $\mathbf{0} = g(\mathbf{0})$, there exists $\delta_3 > 0$ so that if $0 < ||\mathbf{y}|| < \delta_3$, then

$$\frac{||f(\mathbf{y}) - A\mathbf{y}}{||\mathbf{y}||} < \epsilon/2\beta.$$

This implies that

$$||f(\mathbf{y}) - A\mathbf{y}|| < (\epsilon/2\beta)||\mathbf{y}||$$

Pieces 2 and 3 imply: if $0 < \mathbf{x} < \min(\delta_2, \delta_3)$, setting $\mathbf{y} = g(\mathbf{x})$ we have

$$||f(g(\mathbf{x}) - Ag(\mathbf{x})|| < (\epsilon/2\beta)||g(\mathbf{x})|| < (\epsilon/2\beta)\beta||\mathbf{x}||.$$

Consequently, if $0 < \mathbf{x} < \min(\delta_2, \delta_3)$, we have

$$\frac{||f(g(\mathbf{x})) - Ag(\mathbf{x})}{||\mathbf{x}||} < \epsilon/2.$$

Piece 1 implies: if $0 < \mathbf{x} < \delta_1$, then

$$\frac{\alpha||g(\mathbf{x}) - B\mathbf{x}||}{||\mathbf{x}||} < \epsilon/2.$$

We conclude that if $0 < ||\mathbf{x}|| < \delta = \min(\delta_1, \delta_2, \delta_3)$ then

$$\begin{aligned} ||f \circ g(\mathbf{x}) - AB\mathbf{x}||/||\mathbf{x}|| &\leq \\ ||f(g(\mathbf{x})) - Ag(\mathbf{x})||/||\mathbf{x}|| + \alpha||g(\mathbf{x}) - B\mathbf{x}||/||\mathbf{x}|| &< \\ \epsilon/2 + \epsilon/2 &= \epsilon \end{aligned}$$

as desired. $\square$

5. Parameterized Curves

A function $\mathbf{x}\colon \mathbb{R} \to \mathbb{R}^n$ traces out a path in $\mathbb{R}^n$. The set of points

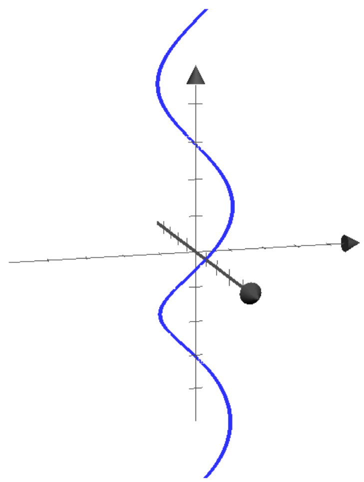
$$\{\mathbf{a} \in \mathbb{R}^n : \mathbf{x}(t) = \mathbf{a} \text{ for some } t \in \mathbb{R}\}$$

is called the **image** of the path $\mathbf{x}$. If $\mathbf{x}$ is one-to-one (so that no place on the image occurs at more than one time) we say that $\mathbf{x}$ is a **parameterization** of its image. We also allow the domain to be a subset of $\mathbb{R}$ rather than all of $\mathbb{R}$.

5.1. Some important examples.

Example 5.1. The path $\mathbf{x_1}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ for $t \in [0, 2\pi]$ is a parameterization of the unit circle in$\mathbb{R}^2$. The path $\mathbf{x_2}(t) = \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}$ for $t \in [0, 2\pi]$ also has image the unit circle but is not a parameterization of it since it travels through each point on the unit circle twice. The path $\mathbf{x_3}(t) = \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}$ for $t \in [0, \pi]$ is parameterization of the unit circle. The path $\mathbf{x_3}$ is different from the path $\mathbf{x_1}$ since it travels around the circle twice as fast.

Example 5.2. The path $\mathbf{x}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$ is a parameterization of a helix in $\mathbb{R}^3$ that winds around the z-axis.



Example 5.3. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function. Then $\mathbf{x}(t) = (t, f(t))$ is a parameterization of the graph of f in $\mathbb{R}^2$. The path $\mathbf{y}(t) = (-t, f(-t))$ is a parameterization of the graph that travels in the opposite direction.

Example 5.4. Suppose that $\mathbf{v}$ and $\mathbf{w}$ are distinct vectors in $\mathbb{R}^n$. Then $\mathbf{x}(t) = t\mathbf{v}+(1-t)\mathbf{w}$ is a parameterization of the line through $\mathbf{v}$ and $\mathbf{w}$. Restricting $\mathbf{x}$ to $t \in [0, 1]$ is a parametrization of the line segment joining $\mathbf{v}$ and $\mathbf{w}$.

Example 5.5. Suppose that $\mathbf{v}$ and $\mathbf{w}$ are distinct vectors in $\mathbb{R}^n$. Then $\mathbf{x}(t) = \mathbf{v} + t\mathbf{w}$ is a parameterization of the line through $\mathbf{v}$ that is parallel to the vector $\mathbf{w}$.

5.2. **Velocity and Acceleration.** If $\mathbf{x}\colon \mathbb{R} \to \mathbb{R}^n$ is a differentiable parameterized curve, its **velocity** is $\mathbf{x}'(t) = D\mathbf{x}(t)$ and its acceleration is $\mathbf{v}'(t) = D\mathbf{v}(t)$. The **speed** of $\mathbf{x}$ is $||\mathbf{x}'(t)||$.

Example 5.6. Find $\mathbf{v}(t)$ and $\mathbf{a}(t)$ for the curve $\mathbf{x}(t) = (t, t\sin(t), t\cos(t))$. Also find the speed of $\mathbf{x}(t)$ at time t.

Solution:

$$\begin{aligned} \mathbf{v}(t) &= (1, \sin(t) + t\cos(t), \cos(t) - t\sin(t)) \\ ||\mathbf{v}(t)|| &= \sqrt{1 + \sin(t)\cos(t) - t^2\sin(t)\cos(t) - t\sin^2(t) + t\cos^2(t)} \\ \mathbf{a}(t) &= (0, 2\cos(t) - t\sin(t), -2\sin(t) - t\cos(t)) \end{aligned}$$

The next theorem should not be surprising.

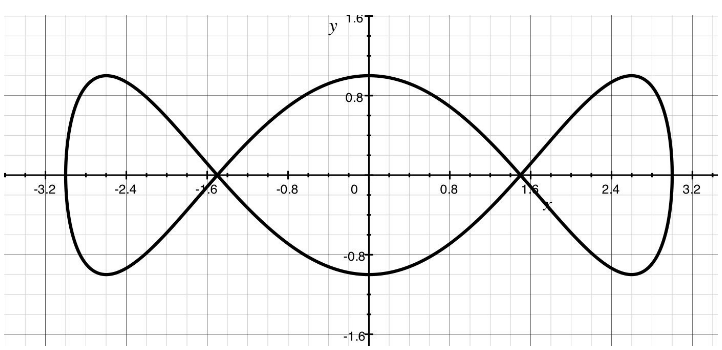
Theorem 5.7. Suppose that $\mathbf{x}\colon \mathbb{R} \to \mathbb{R}^n$ is differentiable. Then $\mathbf{x}'(t_0)$ is parallel to the line tangent to the curve $\mathbf{x}(t)$ at t_0.

Proof. We consider only $n = 2$; for $n > 2$, the proof is nearly identical. A vector parallel to the tangent line to $\mathbf{x}(t)$ at $t = t_0$ can be obtained as in 1-variable calculus:

$$\begin{aligned} \text{tangent vector} &= \lim_{\Delta t \to 0} \big(\mathbf{x}(t_0 + \Delta t) - \mathbf{x}(t_0)\big)/\Delta t \\ &= \lim_{\Delta t \to 0} \Big(\big(x(t_0 + \Delta t), y(t_0 + \Delta t)\big) - \big(x(t_0), y(t_0)\big)\Big)/\Delta t \\ &= \lim_{\Delta t \to 0} \left(\tfrac{x(t_0+\Delta t) - x(t_0)}{\Delta t}, \tfrac{y(t_0+\Delta t) - y(t_0)}{\Delta t}\right) \\ &= \left(\lim_{\Delta t \to 0} \tfrac{x(t_0+\Delta t) - x(t_0)}{\Delta t}, \lim_{\Delta t \to 0} \tfrac{y(t_0+\Delta t) - y(t_0)}{\Delta t}\right) \\ &= (x'(t), y'(t)) \\ &= \mathbf{x}'(t) \end{aligned}$$

□

Example 5.8. Let $\mathbf{x}(t) = (3\cos(2t), \sin(6t))$. The image of $\mathbf{x}$ for $t \in [-6\pi, 6\pi]$ is drawn below. Find the equations of the tangent lines at the point $(-1.5, 0)$.



Solution: The point $(-1.5, 0)$ is crossed by $\mathbf{x}$ at $t_1 = \pi/3$ and at $t_2 = 2\pi/3$. The derivative of $\mathbf{x}$ is

$$\mathbf{x}'(t) = (-6\sin(2t), 6\cos(6t)).$$

At t_1, we have:

$$\mathbf{x}'(t_1) = (-6\sin(2\pi/3), 6\cos(2\pi)) = (-3\sqrt{3}, 6).$$

Thus, one of the tangent lines has parameterization:

$$L_1(t) = t(-3\sqrt{3}, 6) + (-1.5, 0).$$

At t_2, we have:

$$\mathbf{x}'(t_2) = (3\sqrt{3}, 6).$$

Thus, the other tangent line has a parameterization:

$$L_2(t) = t(3\sqrt{3}, 6) + (-1.5, 0).$$

5.3. **Tangent space coordinates.** In the previous section we saw that if $\mathbf{x}$ was a parameterized curve, then $\mathbf{x}'(t)$ is a vector *parallel* to the tangent line to the image of $\mathbf{x}$ at t. It would be much better to base our derivative vector at the point $\mathbf{x}(t)$. We can do this if we change coordinate systems.

Here is the idea:

Example 5.9. Let $\mathbf{x}(t) = (\cos t, \sin t)$ and let $t_0 = (\pi/4, \pi/4)$. Notice that $\mathbf{x}'(t_0) = (1/\sqrt{2}, 1/\sqrt{2})$. If an object's position at time t seconds is given by $\mathbf{x}(t)$ and if at time t_0 all forces stop acting on the object then 1 second later, the object will be at the position given by $\mathbf{x}(t_0) + \mathbf{x}'(t_0)$. That is, $\mathbf{x}'(t_0)$ denotes the direction the object will travel starting at $\mathbf{x}(t_0)$. It would be convenient to represent $\mathbf{x}(t_0)$ by a vector with tail at $\mathbf{x}(t_0)$ and head at $\mathbf{x}(t_0) + \mathbf{x}'(t_0)$.

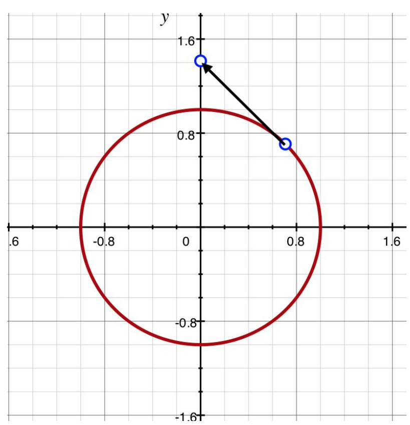


FIGURE 1.

To do this to each point $\mathbf{p} \in \mathbb{R}^n$ we associate a "tangent space" $T_{\mathbf{p}}$. This is simply a copy of $\mathbb{R}^n$ such that $\mathbf{p}$ corresponds to the origin of $T_{\mathbf{p}}$. In $\mathbb{R}^2$, the standard basis vectors are denoted $\mathbf{i}$ and $\mathbf{j}$. In $\mathbb{R}^3$ the standard basis vectors are denoted $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. We usually think of $T_{\mathbf{p}}$ as an alternative coordinate system for $\mathbb{R}^n$ which is positioned so that $\mathbf{p} \in \mathbb{R}^n$ is at the origin.

Example 5.10. If $\mathbf{p} = (1, 3)$ and if $(2, 5) \in T_{\mathbf{p}}$ then $(2, 5)$ corresponds to the point $(1, 3) + (2, 5) = (3, 8)$ in $\mathbb{R}^2$.

We think of $T_{\mathbf{p}}$ as the set of directions at $\mathbf{p}$.

Example 5.11. Let $\mathbf{x}(t) = (\cos t, \sin t)$ and let $t_0 = \pi/6$. Suppose that an object is following the path $\mathbf{x}(t)$ and that at time t_0 all forces stop acting on the object. Then the direction in which the object will head is

$$\mathbf{x}'(t_0) = (-\sin \pi/6, \cos \pi/6) = (-1/2, \sqrt{3}/2).$$

That is, the object will travel $1/2$ units to the left of $\mathbf{x}(t_0)$ and $\sqrt{3}/2$ units up from $\mathbf{x}(t_0)$ in 1 second.

Put another way, the point $\mathbf{x}(t_0) + \mathbf{x}'(t_0)$ is the same as the point $\mathbf{x}'(t_0) \in T_{\mathbf{x}(t_0)}$.

Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{p} \in \mathbb{R}^n$. Then $L\colon T_p \to T_{f(\mathbf{p})}$ defined by

$$L(\mathbf{x}) = Df(\mathbf{p})\mathbf{x}$$

is a linear map between tangent spaces.

Example 5.12. Let $\mathbf{p} = (1, 2) \in \mathbb{R}^2$ and let $f(\mathbf{x}) = (1/4)(x^2 + y^2, x^2 - y^2)$ for all $\mathbf{x} = (x, y)$. Let $\mathbf{v} = (-2, 3) \in T_{\mathbf{p}}$. Sketch the point $Df(\mathbf{p})\mathbf{v} \in T_{f(\mathbf{p})}$.

Solution: Compute:

$$Df(x, y) = \begin{pmatrix} x/2 & y/2 \\ x/2 & -y/2 \end{pmatrix}.$$

So that

$$Df(\mathbf{p}) = \begin{pmatrix} 1/2 & 1 \\ 1/2 & -1 \end{pmatrix}.$$

Thus,

$$Df(\mathbf{p})\mathbf{v} = \begin{pmatrix} 1/2 & 1 \\ 1/2 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}.$$

In $\mathbb{R}^2$, we plot $Df(\mathbf{p})\mathbf{v}$ by starting at $f(\mathbf{p}) = (5/4, -3/4)$ and then travel over 2 and down 4. See Figure 2.

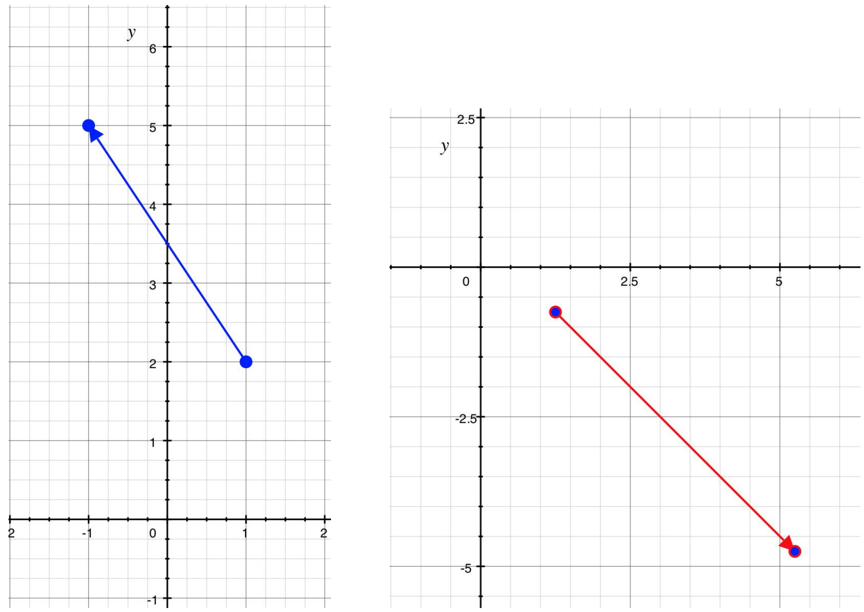


FIGURE 2. On the left is an arrow representing $\mathbf{v} \in T_{\mathbf{p}}$. On the right is an arrow representing $Df(\mathbf{p})\mathbf{v}$ in $T_{f(\mathbf{p})}$.

Example 5.13. Suppose that a circle of radius ρ cm rolls along level ground so that the center of the circle is moving at 1 cm/sec. At time $t = 0$, the center of the circle is at $(0, 0)$ and the top of the circle is a point $P = (0, \rho)$. As the circle rolls, the point P traces out a curve $\mathbf{x}(t)$ (with $P = \mathbf{x}(0)$). Find an equation for $\mathbf{x}(t)$.

Solution: Let $\mathbf{c}(t)$ denote the center of the circle at time t. The circumference of the circle is $2\pi\rho$ and so the circle makes one complete rotation in $2\pi\rho$ sec. At time t, the line segment joining $\mathbf{c}(t)$ to $\mathbf{x}(t)$ makes an angle of $-t/\rho + \pi/2$ with the horizontal. That is, in $T_{\mathbf{c}(t)}$, $\mathbf{x}(t)$ is represented by the point $(\rho\cos(-t/\rho + \pi/2), \rho\sin(-t/\rho + \pi/2))$. Thus, with respect to the standard coordinates on $\mathbb{R}^2$:

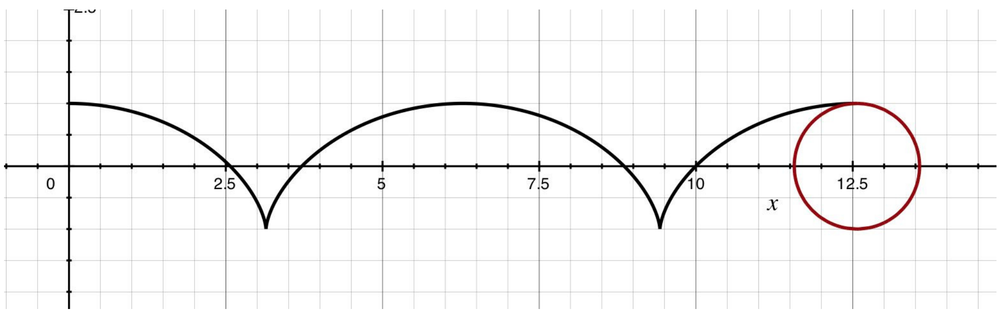
$$\mathbf{x}(t) = \mathbf{c}(t) + \begin{pmatrix} \rho\cos(-t/\rho + \pi/2) \\ \rho\sin(-t/\rho + \pi/2) \end{pmatrix}.$$

Since

$$\mathbf{c}(t) = t\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

we have

$$\mathbf{x}(t) = \begin{pmatrix} t + \rho\cos(-t/\rho + \pi/2) \\ \rho\sin(-t/\rho + \pi/2) \end{pmatrix}.$$

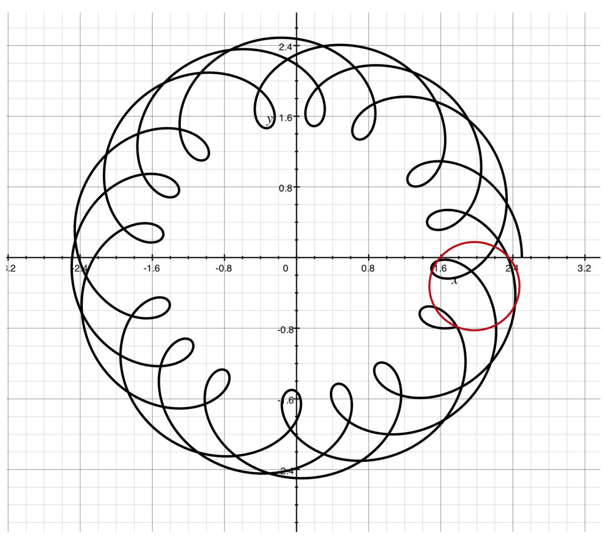


Question: Is the cycloid a differentiable curve?

Example 5.14. Suppose that a circle C of radius r is moving so that the center of C, $\mathbf{c}$ traces out the path $(R\cos(t), R\sin(t))$. As C moves, it rotates counterclockwise so that it completes k revolutions per second. Suppose that E is the East pole of C at time 0. What path does P trace out?

Solution: In $T_{\mathbf{c}(t)}$, E has coordinates $(r\cos 2\pi kt, r\sin 2\pi kt)$. Thus in $\mathbb{R}^2$ coordinates, E has position

$$\mathbf{x}(t) = \mathbf{c}(t) + (r\cos t, r\sin t) = (R\cos t + r\cos 2\pi kt, R\sin t + r\sin 2\pi kt).$$



5.4. **Intrinsic vs. Extrinsic.**

Example 5.15. Consider the parameterizations

$$\mathbf{x}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \qquad \text{for } 0 \leq t \leq 2\pi$$

and

$$\mathbf{y}(t) = \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} \qquad \text{for } 0 \leq t \leq \pi.$$

Both are parameterizations of the unit circle. To use either of them to study the unit circle we need to develop properties of parameterizations that depend only on the underlying curve and not on the parameterization chosen. Such properties are called "intrinsic" properties of the curve.

Definition 5.16. A function $h\colon [c,d] \to [a,b]$ is a **change of coordinates function** if it is a C^1 bijection. Often we will also require h to have the property that for all t, $h'(t) \neq 0$.

(Recall: C^1 means that its derivative always exists and is continuous. A "bijection" is a function that is both one-to-one and onto.)

Example 5.17. Define $h(t)\colon [0,\pi] \to [0,2\pi]$ by $h(t) = 2t$. The function h is a change of coordinates function.

Example 5.18. Define $h\colon [0,1] \to [0,1]$ by $h(t) = t^3$. Then h is a change of coordinates function, but its inverse $h^{-1}(t) = \sqrt[3]{t}$ is not a change of coordinates function because it is not differentiable at $t = 0$.

Example 5.19. Find a change of coordinates function $h\colon [0,1] \to [0,1]$ such that $h(0) = 1$ and $h(1) = 0$. (This is an example of an "orientation-reversing" change of coordinates function.

Solution: Sketch the x and y axes. Any function that is monotonically decreasing from the point $(0,1)$ to the point $(1,0)$ will work. The function $h(t) = -t + 1$ is one such function.

Notice that if h is a change of coordinates function with $h'(t) \neq 0$ for any t, then either $h'(t) > 0$ for all t or $h'(t) < 0$ for all t (by the intermediate value theorem).

Definition 5.20. A change of coordinates function $h\colon [c,d] \to [a,b]$ is **orientation-preserving** if $h'(t) > 0$ for all t and is **orientation-reversing** if $h'(t) < 0$ for all t. Notice that if h is orientation-preserving then $h(c) = a$ and $h(d) = b$ while if h is orientation-reversing then $h(c) = b$ and $h(d) = a$.

Definition 5.21. If $\mathbf{x}\colon [a,b] \to \mathbb{R}^n$ and $\mathbf{y}\colon [c,d] \to \mathbb{R}^n$ are paths, we say that $\mathbf{y}$ is a **reparameterization** of $\mathbf{x}$ if there exists a change of coordinates function $h\colon [c,d] \to [a,b]$ such that $\mathbf{y} = \mathbf{x} \circ h$. If h is orientation preserving, we say that $\mathbf{y}$ is an **orientation-preserving reparameterization** of $\mathbf{x}$. If h is orientation reversing, we say that $\mathbf{y}$ is an **orientation-reversing** reparameterization of $\mathbf{x}$.

The key point is that: *reparameterizing a curve is changing the speed and possibly the direction that we walk along the curve.* Intuitively, the change-of-coordinates function h tells us how to speed up or slow down as we traverse that path laid down by $\mathbf{x}$. If $\mathbf{y}$ is an orientation-preserving reparameterization of $\mathbf{x}$, it traces out the path in the same direction that $\mathbf{x}$ did, otherwise it traces the path out in the opposite direction.

Example 5.22. Let $\mathbf{x}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$ for $t \in [0, 5]$. Let $\mathbf{y}(t) = \begin{pmatrix} 9t^2 \\ 6t \end{pmatrix}$ for $t \in [0, 5/3]$.

The path $\mathbf{y}$ is an orientation reparameterization of $\mathbf{x}$ by the change of coordinates function $h(t) = 3t$.

Example 5.23. Let $\mathbf{x}(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$ and let $\mathbf{y}(t) = (\cos 3t, \sin 3t)$ for $t \in [0, 2\pi]$. Then $\mathbf{y}$ is not a reparameterization of $\mathbf{x}$ since $\mathbf{x}$ traverses the unit circle once, but $\mathbf{y}$ traverses it three times.

Example 5.24. Let $\mathbf{x}(t) = \begin{pmatrix} t \\ \cos t \\ \sin t \end{pmatrix}$ for $t \in [\pi, 2\pi]$. Let $\mathbf{y}(t) = \begin{pmatrix} t^3 \\ \cos t^3 \\ \sin t^3 \end{pmatrix}$ for $t \in [\sqrt[3]{\pi}, \sqrt[3]{2\pi}]$.

The path $\mathbf{y}$ is an orientation-preserving reparameterization of $\mathbf{x}$ using the change of coordinates function $h(t) = t^3$ for $t \in [\sqrt[3]{\pi}, \sqrt[3]{2\pi}]$.

Example 5.25. Let G be the graph of a function $y = f(x)$ for $x \in [a, b]$. Find two parameterizations, with opposite orientations, of G.

One parameterization is $\mathbf{x}(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix}$ for $t \in [a, b]$. A second one is $\mathbf{y}(t) = \begin{pmatrix} -t \\ f(-t) \end{pmatrix}$ for $t \in [-b, -a]$. Since they are related by the change of coordinates function $h(t) = -t$ which has $h'(t) = -1$, the curves have the opposite orientations.

Definition 5.26. A quantity is **intrinsic** if it does not change under orientation preserving reparameterization. A quantity is **intrinsic to oriented curves** if it does not change under orientation preserving reparameterization.

Example 5.27. If $\mathbf{x}\colon [a, b] \to \mathbb{R}^n$ is a path, its derivative and speed are not intrinsic, since we can walk the same path at a different speed.

Example 5.28. Let $\mathbf{x}\colon [a, b] \to \mathbb{R}^n$ be a C^1 path such that $||\mathbf{x}'(t)|| \neq 0$ for any t. Then the unit tangent vector

$$\mathbf{T}_{\mathbf{x}}(t) = \frac{\mathbf{x}'}{||\mathbf{x}'||}$$

is intrinsic to the oriented curve.

To see this, suppose that $\mathbf{y}\colon [c,d] \to \mathbb{R}^n$ is another C^1 path such that $||\mathbf{y}'(t)|| \neq 0$ for any t and suppose that $h\colon [c,d] \to [a,b]$ is an orientation preserving change of coordinate function so that $\mathbf{y} = \mathbf{x} \circ h$. We need to show that $\mathbf{T}_{\mathbf{y}}(t) = \mathbf{T}_{\mathbf{x}}(h(t))$ for all t.

By the chain rule,

$$\mathbf{y}'(t) = \mathbf{x}'(h(t))h'(t).$$

Notice that since h is orientation preserving, $h'(t) = |h'(t)|$ for all t. Thus,

$$\begin{aligned}
\mathbf{T}_{\mathbf{y}}(t) &= \tfrac{\mathbf{y}'(t)}{||\mathbf{y}'(t)||} \\
&= \tfrac{\mathbf{x}'(h(t))h'(t)}{||\mathbf{x}'(h(t))h'(t)||} \\
&= \tfrac{\mathbf{x}'(h(t))h'(t)}{||\mathbf{x}'(h(t))||\,|h'(t)|} \\
&= \tfrac{\mathbf{x}'(h(t))h'(t)}{||\mathbf{x}'(h(t))||\,h'(t)} \\
&= \tfrac{\mathbf{x}'(h(t))}{||\mathbf{x}'(h(t))} \\
&= \mathbf{T}_{\mathbf{x}}(t)
\end{aligned}$$

Example 5.29. Here is an informal example that we will develop in the next section.

If $\mathbf{x}\colon [a,b] \to \mathbb{R}^n$ is a curve that, as a function, is one-to-one, we can define its length to be the limit of the lengths of piecewise linear approximations to the image of $\mathbf{x}$. Since this does not depend on the parameterization this length that we calculate is intrinsic to $\mathbf{x}$.

5.5. **Arc length.** Suppose that $\mathbf{x}\colon [a,b] \to \mathbb{R}$ is a C^1 curve. We wish to find the length of $\mathbf{x}$. The formula is

Theorem 5.30. The arc length of $\mathbf{x}$ is

$$\int_a^b ||\mathbf{x}'(t)||\, dt.$$

Arc length is often denote by

$$\int_{\mathbf{x}} ds$$

where

$$ds = ||\mathbf{x}'||dt$$

Example 5.31. Let $\mathbf{x}(t) = (t^2, 2t^2)$ for $t \in [0,1]$. Then

$$||\mathbf{x}'(t)|| = ||(2t, 4t)|| = \sqrt{4t^2 + 16t^2} = 2t\sqrt{5}.$$

The arclength of $\mathbf{x}$ is

$$\int_{\mathbf{x}} ds = \int_0^1 2t\sqrt{5}\, dt = t^2\sqrt{5}\Big|_0^1 = \sqrt{5}.$$

Example 5.32. Let $\mathbf{x}(t) = (t, t^2)$ for $t \in [0,1]$. Then

$$\int_{\mathbf{x}} ds = \int_0^1 \sqrt{1+4t^2}\, dt \approx 1.47894$$

Here is why the formula for arclength is what it is. For convenience, we assume that $n = 2$.

Partition $[a,b]$ into n subintervals $[t_{i-1}, t_i]$ for $1 \leq i \leq n$, each of length $\Delta t = (b-a)/n$. Joining the points $\mathbf{x}(t_{i-1})$ and $\mathbf{x}(t_i)$ by straight lines creates a polygonal approximation P_n to the image of $\mathbf{x}$. The length of the polygonal path is:

$$\text{length}(P_n) = \sum_{i=1}^n ||\mathbf{x}(t_i) - \mathbf{x}(t_{i-1})||.$$

We *define* the **arc length** of $\mathbf{x}$ to be

$$L = \int_{\mathbf{x}} ds = \lim_{n\to\infty} \sum_{i=1}^n ||\mathbf{x}(t_i) - \mathbf{x}(t_{i-1})||.$$

Now suppose that $\mathbf{x}(t) = (x(t), y(t))$. Both x and y are C^1 functions. Notice that if we replace our current polygonal approximation with a polygonal approximation have vertices $(x(t_i^*), y(t_i^{**}))$, with $t_i^*, t_i^{**} \in [t_{i-1}, t_i]$, we will still have:

$$L = \int_{\mathbf{x}} ds = \lim_{n\to\infty} \sum_{i=1}^n ||(x(t_i^*), y(t_i^{**})) - (x(t_{i-1}^*), y(t_{i-1}^{**}))||.$$

Here's how to choose the values t_i^* and t_i^{**}. By the mean value theorem (remember that?) There exists $t_i^*, t_i^{**} \in [t_{i-1}, t_i]$ so that

$$\begin{array}{rcccl} x(t_i^*) &=& x'(t_i^*)(t_i - t_{i-1}) &=& x'(t_i^*)\Delta t \\ y(t_i^{**}) &=& y'(t_i^{**})(t_i - t_{i-1}) &=& y'(t_i^{**})\Delta t \end{array}$$

Thus,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\left(x'(t_i^*)^2 + y'(t_i^{**})^2\right.}\Delta t = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_a^b ||\mathbf{x}'(t)||\, dt.$$

We can also compute the arc length of paths which are piecewise C^1. These paths must be composed of a finite number of pieces.

Example 5.33. Compute the length of the curve $\mathbf{x}\colon [0,2] \to \mathbb{R}$ defined by:

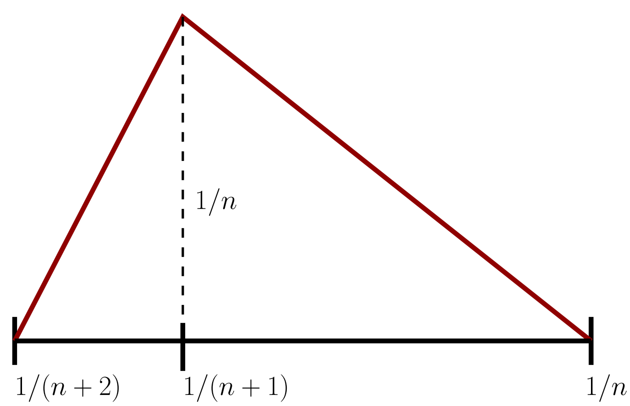
$$\mathbf{x}(t) = \left\{ \begin{array}{cc} (t, t^2) & \text{if } 0 \le t \le 1 \\ (t, (2-t)^2) & \text{if } 1 \le t \le 2 \end{array} \right\}$$

Solution: Let $\mathbf{x_1}(t) = \mathbf{x}(t)$ for $0 \le t \le 1$ and let $\mathbf{x}_2(t) = \mathbf{x}(t)$ for $1 \le t \le 2$. Then

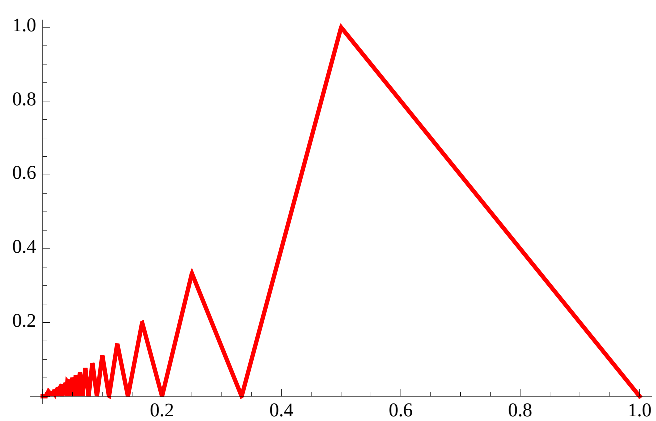
$$\begin{array}{rcl} \int_{\mathbf{x}} ds & = & \int_{\mathbf{x_1}} ds + \int_{\mathbf{x_2}} ds \\ & = & \int_0^1 \sqrt{1+4t^2}\, dt + \int_1^2 \sqrt{1+4(2-t)^2} \\ & \approx & 2.95789 \end{array}$$

The following example shows that it is possible for a "finite" curve to have infinite length.

Example 5.34. We will specify the graph of the curve $f(x)$. On the interval $[\frac{1}{n+2}, \frac{1}{n}]$ erect a tent consisting of two straight lines with the bottoms of the lines on the x axis and the top of the tent at the point $(\frac{1}{n+1}, \frac{1}{n})$. See the figure below:



Do this for each odd value of n, achieving the following graph:



If you want an equation for $f(x)$ do the following:

Begin by defining

$$g_n(x) = \left\{ \begin{array}{cl} 0 & \text{if } x < \frac{1}{n+2} \\ \frac{1}{n\left(\frac{1}{n+1} - \frac{1}{n}\right)}(x - \frac{1}{n+2}) & \text{if } \frac{1}{n+2} \leq x \leq \frac{1}{n+1} \\ \\ \frac{-1}{n\left(\frac{1}{n} - \frac{1}{n+1}\right)}(x - \frac{1}{n}) & \text{if } \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0 & \text{if } x > \frac{1}{n} \end{array} \right\}$$

Then define

$$f(x) = \sum_{n=0}^{\infty} g_{2n+1}(x).$$

Notice that $g_{2n+1}(x) \neq 0$ only if $x \in [\frac{1}{2n+3}, \frac{1}{2n+1}]$. Thus, the sum defining $f(x)$ has only one term which is not zero.

Let's show that the length of the graph of f is infinite. To do this, consider the line segment in the interval $[\frac{1}{n+1}, \frac{1}{n}]$ for an odd value of n. This line segment has length

$$L = \sqrt{(\frac{1}{n})^2 + (\frac{1}{n} - \frac{1}{n+1})^2} = \sqrt{\frac{2}{n^2} + \frac{1}{(n+1)^2} - \frac{2}{n(n+1)}}.$$

Some algebra shows that $L \geq \frac{1}{n}$ Similarly, the line segment in the interval $[\frac{1}{n+2}, \frac{1}{n+1}]$ has length at least $1/(n+1)$. Consequently, the length of the graph of f is at least

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

It is well known that this is the harmonic series which diverges to infinity.

The text gives an example of a function $f\colon [0,1] \to [-1,1]$ which is differentiable on $(0,1]$ but whose graph has infinite arclength. An example similar to that one could be constructed from our example by rounding the points of the graph above.

Theorem 5.35 (Arc length is intrinsic)**.** Suppose that $\mathbf{x}\colon [a,b] \to \mathbb{R}^n$ and $\mathbf{y}\colon [c,d] \to \mathbb{R}^n$ are C^1 curves and that $\mathbf{y}$ is a reparameterization of $\mathbf{x}$. Then the length of $\mathbf{y}$ is equal to the length of $\mathbf{x}$.

Proof. Since $\mathbf{y}$ is a reparameterization of $\mathbf{x}$, there exists a change-of-coordinates function $h\colon [c,d] \to [a,b]$ such that $\mathbf{y} = \mathbf{x} \circ h$. By the chain rule we have:

$$\mathbf{y}'(t) = \mathbf{x}'(h(t))h'(t).$$

Taking magnitudes gives:

$$||\mathbf{y}'(t)|| = ||\mathbf{x}'(h(t))||\,|h'(t)|.$$

Case 1: h is orientation-preserving. In this case, $|h'(t)| = h'(t)$. Then, by definition, the length of $\mathbf{y}$ is:

$$\begin{aligned} L(\mathbf{y}) &= \int_c^d ||\mathbf{y}'(t)||\,dt \\ &= \int_c^d ||\mathbf{x}'(t)||h'(t)\,dt. \end{aligned}$$

Let $u = h(t)$. Then $du = h'(t)\,dt$ and $u(c) = a$ and $u(d) = b$ since h is orientation preserving. Thus, substitution shows that:

$$\int_c^d ||\mathbf{x}'(t)||h'(t)\,dt = \int_a^b ||\mathbf{x}(u)||\,du.$$

This latter integral is exactly the length of $\mathbf{x}$.

Case 2: h is orientation-reversing.

This case is left to the reader. It follows from the observations that $|h'(t)| = -h'(t)$ and $h(c) = b$ and $h(d) = a$. $\square$

Being able to calculate arclength is not just an end-in-itself. It also gives us a useful way of reparameterizing so that we always travel at unit speed (eg. 1 meter/sec.) This reparameterization is called **reparameterization by arclength**. If a curve $\mathbf{x}\colon [a,b] \to \mathbb{R}^n$ has the property that for all t, $||\mathbf{x}'(t)|| = 1$, we say that $\mathbf{x}$ is **parameterized by arclength**. Here's how to do it:

Suppose that $\mathbf{x}\colon [a,b] \to \mathbb{R}$ is C^1 and that $||\mathbf{x}'(t)|| > 0$ for all $t \in [a,b]$. Define $s\colon [a,b] \to [0,L]$ by

$$s(t) = \int_a^t ||\mathbf{x}'(\tau)||\,d\tau.$$

Notice that s is a strictly increasing C^1 function and so is an orientation preserving bijection $[a,b] \to [0,L]$. By the fundamental theorem of Calculus, $s'(t) = ||\mathbf{x}'(t)||$ so we often write $ds = ||\mathbf{x}'(t)||dt$.

Furthermore, its inverse function $s^{-1}\colon [a,b] \to [a,L]$ is also strictly increasing bijection. Define $\mathbf{y}(t) = \mathbf{x} \circ s^{-1}$. The function s measures the distance travelled from time a to time t using the path $\mathbf{x}$. Composing $\mathbf{x}$ with s^{-1} makes it so that $\mathbf{x}$ travels at one unit of distance per unit of time. (Like how driving at 60 mph means that you travel at 1 mile per minute.)

For the record:

How to reparameterize by arclength, given $\mathbf{x}\colon [a,b] \to \mathbb{R}^n$ such that $||\mathbf{x}'(t)|| \neq 0$.

(1) Find $s\colon [a,b] \to [0,L]$ defined by

$$s(t) = \int_a^t ||\mathbf{x}'(\tau)||\,d\tau.$$

(2) Find the inverse function $s^{-1}\colon [0,L] \to [a,b]$. Do this by setting $\sigma = s(t)$ and solving for t.
(3) Define $\mathbf{y}(\sigma) = \mathbf{x}(s^{-1}(\sigma))$. That is, $\mathbf{y} = \mathbf{x} \circ s^{-1}$. The curve $\mathbf{y}$ is the reparameterization of $\mathbf{x}$ by arclength, as we show after some examples.

Example 5.36. Reparameterizing $\mathbf{x}(t) = \begin{pmatrix} t \\ 3t \end{pmatrix}$ for $t \in [0, 2]$ by arclength.

Solution: Notice that $||\mathbf{x}'(t)|| = \sqrt{10}$. Thus,

$$s(t) = \int_0^t \sqrt{10}\, d\tau = \sqrt{10}t.$$

Let $\sigma = \sqrt{10}t$ and solve for t to find that:

$$s^{-1}(\sigma) = t = \sigma/\sqrt{10}.$$

Thus,

$$\mathbf{y}(\sigma) = \mathbf{x}(s^{-1}(\sigma)) = \begin{pmatrix} \sigma/\sqrt{10} \\ 3\sigma/\sqrt{10} \end{pmatrix}$$

is a reparameterization of $\mathbf{x}$ by arclength.

Example 5.37. Let $\mathbf{x}(t) = \begin{pmatrix} t \\ t \\ (2/3)t^{3/2} \end{pmatrix}$ for $t \in [1, 10]$. Reparameterize $\mathbf{x}$ by arclength.

Solution: Notice that $\mathbf{x}'(t) = (1, 1, t^{1/2})$ and that $||\mathbf{x}'(t)|| = \sqrt{2+t}$. Thus,

$$\sigma = s(t) = \int_1^t \sqrt{2+\tau}\, d\tau = (2/3)(2+t)^{3/2}.$$

Solving for t we get:

$$s^{-1}(\sigma) = t = (3\sigma/2)^{2/3} - 2.$$

We plug into $\mathbf{x}$ to get:

$$\mathbf{y}(\sigma) = \mathbf{x}(s^{-1}(\sigma) = \begin{pmatrix} (3\sigma/2)^{2/3} - 2 \\ (3\sigma/32)^{2/3} - 2 \\ (2/3)((3\sigma/2)^{3/2} - 2)^{(3/2)} \end{pmatrix}.$$

Next we prove that the steps to reparameterization work, and then we do some other examples.

Lemma 5.38. Assume that $\mathbf{x}$ is a C^1 curve defined on $[a, b]$ such that for all t, $||\mathbf{x}'(t)|| \neq 0$. Let $\mathbf{y}$ be the reparameterization of $\mathbf{x}$ by arc length. Then for all t, $||\mathbf{y}'(t)|| = 1$ and the length of $\mathbf{y}$ on the interval $[0, t]$ is t.

Proof. Notice that:

$$s'(t) = ||\mathbf{x}'(t)||$$

by the fundamental theorem of Calculus. Also, $\mathbf{y} = \mathbf{x} \circ s^{-1}$ means that $\mathbf{x} = \mathbf{y} \circ s$. Consequently, by the chain rule,

$$||\mathbf{x}'(t)|| = ||\mathbf{y}'(s(t))||\, |s'(t)|$$

Letting $\sigma = s(t)$ and recalling that $s'(t) = ||\mathbf{x}'(t)||$ we get:

$$||\mathbf{x}'(t)|| = ||\mathbf{y}'(\sigma)||\,||\mathbf{x}'(t)||.$$

Thus, since $||\mathbf{x}'(t)|| \neq 0$,

$$||\mathbf{y}'(\sigma)|| = 1.$$

The length of $\mathbf{y}$ on the interval $[0, t]$ is, by definition,

$$\int_0^t ||\mathbf{y}'(\sigma)||\,d\sigma.$$

We see immediately that this equals t. □

Example 5.39. Let $\mathbf{x}(t) = (t^2, 3t^2)$ for $t \in [1, 2]$. Reparameterize $\mathbf{x}$ by arc length.

Answer: By definition,

$$\begin{array}{rcl} s(t) & = & \int_1^t \sqrt{4\tau^2 + 36\tau^2}\,d\tau \\ & = & \int_1^t \sqrt{40}\tau\,d\tau \\ & = & \sqrt{40}(t^2 - 1) \end{array}$$

We need, s^{-1}. Solving the previous equation for t we find:

$$t = \sqrt{1 + s/\sqrt{40}}$$

Thus,

$$s^{-1}(t) = \sqrt{1 + t/\sqrt{40}}$$

To get $\mathbf{y}(t)$ which is the reparameterization of $\mathbf{x}$ by arclength, we plug this in for t in the equation for $\mathbf{x}$, getting:

$$\begin{array}{rcl} \mathbf{y}(t) & = & \mathbf{x} \circ s^{-1}(t) \\ & = & (\left(\sqrt{1 + t/\sqrt{40}}\right)^2, 3\left(\sqrt{1 + t/\sqrt{40}}\right)^2) \\ & = & \left(1 + t/\sqrt{40}, 3(1 + t/\sqrt{40})\right) \end{array}$$

To avoid much of this algebra, we will often simply write $\mathbf{x}(s)$ instead of $\mathbf{x} \circ s^{-1}$. This notation has the potential to be confusing. Thus, in the previous example, the reparameterization of $\mathbf{x}(t) = (t^2, 3t^2)$ by arc length is

$$\mathbf{x}(s) = \left(1 + s/\sqrt{40}, 3(1 + s/\sqrt{40})\right).$$

Example 5.40. Let $\mathbf{x}(t) = (\cos t, \sin t, (2/3)t^{3/2})$ for $t \geq 3$. Find $\mathbf{x}(s)$.

Answer: Compute:

$$||\mathbf{x}'(t)|| = ||(-\sin t, \cos t, t^{1/2})|| = \sqrt{1 + t}.$$

Thus,

$$s = \int_3^t \sqrt{1 + \tau}\,d\tau = (2/3)(1 + t)^{3/2} - (2/3)(1 + 3)^{3/2} = (2/3)(1 + t)^{3/2} - 16/3.$$

Consequently,

$$t = \left(\frac{3(s + 16/3)}{2}\right)^{2/3}$$

Thus,

$$\mathbf{x}(s) = \left(\cos\left(\frac{3(s+16/3)}{2}\right)^{2/3}, \sin\left(\frac{3(s+16/3)}{2}\right)^{2/3}, (2/3)\left(\frac{3(s+16/3)}{2}\right)^{4/3} \right)$$

5.6. **Curvature and the Moving Frame.** In this section we assign an intrinsic tangent space coordinate system to each point $\mathbf{x}$ on the image of a path $\mathbf{x}\colon [a,b] \to \mathbb{R}^3$ called **the moving frame**.

We begin with an important lemma. It can be interpreted as saying that if a path lies on a sphere then its position vector is perpendicular to its tangent vector. This should not be a surprise if we believe that the radius of a circle is perpendicular to the tangent line intersecting it.

Lemma 5.41. Suppose that $\phi\colon [a,b] \to \mathbb{R}^n$ is a C^1 path such that $||\phi(t)||$ is constant. Then, for all t, $\phi(t)$ is perpendicular to $\phi'(t)$.

Proof. Since $||\phi(t)||$ is constant,

$$\phi(t) \cdot \phi(t) = ||\phi(t)||$$

is constant. Thus,

$$\frac{d}{dt}\phi(t) \cdot \phi(t) = 0.$$

By the product rule we have:

$$\phi'(t) \cdot \phi(t) + \phi(t) \cdot \phi'(t) = 0.$$

Since dot product is commutative, this implies that

$$2\phi(t) \cdot \phi'(t) = 0.$$

Consequently,

$$\phi(t) \cdot \phi'(t) = 0$$

as desired. $\square$

We already know that the unit tangent vector $\mathbf{T}$ is intrinsice to an oriented curve, so we take it as one of our coordinate directions in our moving frame. By the previous lemma, its derivative $\mathbf{T}'$ is perpendicular to it. So we define the **unit normal vector** to be

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}.$$

Our third coordinate direction needs to be of unit length and perpendicular to both $\mathbf{T}$ and $\mathbf{B}$. We call it the **unit binormal vector:**

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

Using the unit tangent vector we can also define the **curvature** of a curve $\mathbf{x}$ to be:

$$\kappa(t) = \frac{||\mathbf{T}'(t)||}{||\mathbf{x}'(t)||}.$$

Example 5.42. Find the curvature of a line $\mathbf{x}(t) = t\mathbf{v} + \mathbf{b}$.

Answer: We have

$$\mathbf{T} = \mathbf{x}'/||\mathbf{x}|| = \mathbf{v}/||\mathbf{v}||.$$

Thus, $d\mathbf{T}/dt = \mathbf{0}$ and so $\kappa(t) = 0$.

Example 5.43. The curvature of a circle of radius $r > 0$ is $1/r$ at each point on the circle.

Example 5.44. Let $\phi(t) = (t, at^2)$ be a parameterized curve. Find the curvature of ϕ at $t = 0$.

Answer: We have: $\phi'(t) = (1, 2at)$ and $\mathbf{T} = (1, 2at)/\sqrt{1 + 4a^2t^2}$. Thus,

$$\frac{d}{dt}\mathbf{T} = (0, 2a)/\sqrt{1 + 4a^2t^2} + (1, 2at)(-1/2)(1 + 4a^2t^2)^{-3/2}(8a^2t).$$

Thus,

$$||\phi'(0)|| = 1$$

and

$$||\frac{d}{dt}\mathbf{T}(0)|| = ||(0, 2a)|| = 2a$$

Consequently,

$$\kappa(t) = 2a/1 = 2a$$

Example 5.45. Compute the moving frame and curvature for the path $\mathbf{x}(t) = (\sin t - t\cos t, \cos t + t \sin t, 2)$ with $t \geq 0$.

Answer: We compute:

$$\begin{array}{rcll}
\mathbf{x}'(t) & = & (\cos t - \cos t + t\sin t, -\sin t + \sin t + t\cos t, 0) & = (t\sin t, t\cos t, 0) \\ \hline
||\mathbf{x}'(t)|| & = & \sqrt{t^2\sin^2 t + t^2\cos^2 t} & = t \\ \hline
\mathbf{T} & = & \mathbf{x}'(t)/||\mathbf{x}'(t)|| & = (\sin t, \cos t, 0) \\ \hline
\mathbf{T}' & = & (-\cos t, \sin t, 0) & \\ \hline
||\mathbf{T}'|| & = & 1 & \\ \hline
\mathbf{N} & = & \mathbf{T}'/||\mathbf{T}'|| & = (-\cos t, \sin t, 0) \\ \hline
\kappa & = & ||\mathbf{T}'||/||\mathbf{x}'|| & = 1/t
\end{array}$$

Finally, to compute $\mathbf{B}$ we need the cross product:

$$\mathbf{B} = (\sin t, \cos t, 0) \times (-\cos t, \sin t, 0) = (0, 0, 1).$$

It turns out that

$$\frac{\mathbf{B}'(t)}{||\mathbf{x}'(t)||} = -\tau\mathbf{N}$$

for some scalar function τ, called the **torsion**. If $\tau(t) = 0$ for all t, the Binormal vector is constant and so is the plane perpendicular to it. That plane contains $\mathbf{x}$ and so the torsion measures how much the curve twists out of a plane. If $\tau(t) = 0$ for all t, then the curve lies in a plane.

Example 5.46. Let $\mathbf{x}(t) = \begin{pmatrix} \sin t - t\cos t \\ \cos t + t\sin t \\ t^2 \end{pmatrix}$ for $t > 0$. Calculate $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, κ, and τ for $\mathbf{x}$.

Easy computations show that:

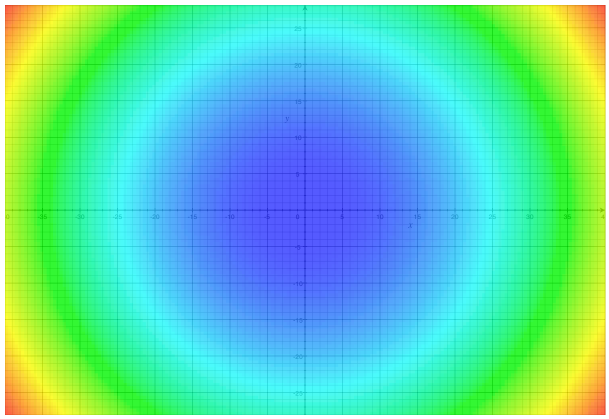
$$\begin{aligned} \mathbf{x}'(t) &= \begin{pmatrix} t\sin t \\ t\cos t \\ 2t \end{pmatrix} \\ ||\mathbf{x}'(t)|| &= t\sqrt{5}. \end{aligned}$$

More computations show:

$$\begin{aligned} \mathbf{T}(t) &= \tfrac{1}{\sqrt{5}} \begin{pmatrix} \sin t \\ \cos t \\ 2 \end{pmatrix} \\ \mathbf{T}'(t) &= \tfrac{1}{\sqrt{5}} \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix} \\ \mathbf{N}(t) &= \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix} \\ \kappa(t) &= \tfrac{1}{5t} \\ \mathbf{B}(t) &= \tfrac{1}{\sqrt{5}} \begin{pmatrix} 2\sin t \\ 2\cos t \\ -1 \end{pmatrix} \\ \mathbf{B}'(t) &= \tfrac{1}{\sqrt{5}} \begin{pmatrix} 2\cos t \\ -2\sin t \\ 0 \end{pmatrix} \\ \mathbf{B}'(t)/||\mathbf{x}'(t)|| &= \tfrac{2}{5t}\mathbf{N}(t) \\ \tau(t) &= -\tfrac{2}{5t}. \end{aligned}$$

.

6.1. Path Integrals of Scalar Fields. A **scalar field** on $\mathbb{R}^n$ is simply a function $f\colon \mathbb{R}^n \to \mathbb{R}$. We think of f as assigning a number $f(\mathbf{x})$ to each point $\mathbf{x}$ in $\mathbb{R}^n$. Below is a depiction of the scalar field $f(x,y) = x^2 + y^2$ on $\mathbb{R}^2$. To a point $(x,y) \in \mathbb{R}^2$, we assign the number $x^2 + y^2$. Points which are assigned small numbers are colored blue and points which are assigned large numbers are colored red.



The following example demonstrates the important idea of integrating a scalar field over a curve.

Example 6.1. Let L be a straight piece of wire in $\mathbb{R}^2$ with endpoints at $(0,0)$ and at $(1,2)$. Suppose that the temperature of the wire at point (x,y) is $f(x,y) = x^2 + y$. Find the average temperature of the wire.

Solution: Break the wire L into little tiny segments, $L_1, \ldots, L_n$ each of length Δs. Since L has a length of $\sqrt{5}$, $\Delta s = \sqrt{5}/n$.

Then the average temperature of L is approximately

$$T_n = \frac{1}{n} \sum_{i=1}^{n} f(\mathbf{x}_\mathbf{i}^*)$$

In fact, the average temperature of L is exactly

$$T = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(\mathbf{x}_\mathbf{i}^*).$$

Recall that $1/n = (\Delta s)/\sqrt{5}$. Thus,

$$T = \lim_{n\to\infty} \frac{1}{\sqrt{5}} \sum_{i=1}^{n} f(\mathbf{x}_\mathbf{i}^*)\Delta s$$

This looks a lot like a limit of Riemann sums, so perhaps we can convert this to a definite integral and use the Fundamental Theorem of Calculus. Before we do that, however, notice that (up to proving that the limit exists) we have a perfectly fine definition of the quantity

$$\text{Ave. value of } f \text{ on } L = \frac{1}{\text{length of } L} \int_L f\, ds.$$

We were able to define this integral without relying on a parameterization of L!

To calculate this, however, we need a parameterization. Suppose that there exists a parameterization $\phi\colon [0, \sqrt{5}] \to \mathbb{R}^2$ of L such that at time t, the distance from $(0, 0)$ to $\phi(t)$ along L is exactly t. That is, "L is parameterized by arc length". Then, $\Delta s = \Delta t = \sqrt{5}/n$ so

$$T = \frac{1}{\sqrt{5}} \lim_{n\to\infty} \sum_{i=1}^{n} f(\phi(t_i^*))\Delta t = \frac{1}{\sqrt{5}} \int_0^{\sqrt{5}} f(\phi(t))\, dt.$$

Exercise: Find a parameterization of L by arclength.

Solution: Define $\hat{\phi}(t) = (t, 2t)$ and define $\phi(t) = \hat{\phi}(t/\sqrt{5})$.

We make the following definition:

Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous and that $\mathbf{x}\colon [a, b] \to \mathbb{R}^n$ is a (piecewise) C^1 path. Define

$$\int_{\mathbf{x}} f\, ds = \int_a^b f(\mathbf{x}(t))||\mathbf{x}'(t)||\, dt.$$

Example 6.2. Let $f(x, y) = x^2 + y$ and $\mathbf{x}(t) = t \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for $0 \le t \le 1$. Then,

$$\begin{aligned} \textstyle\int_{\mathbf{x}} f\, ds &= \textstyle\int_0^1 f(\mathbf{x}(t))|\, ||\vec{x}'(t)||\, dt \\ &= \textstyle\int_0^1 (t^2 + 2t)\sqrt{5}\, dt. \end{aligned}$$

Example 6.3. Let $f(x, y, z) = 1/(xyz)$ and $\mathbf{x}(t) = \begin{pmatrix} \sin t \\ t\cos t \\ t \end{pmatrix}$ for $\pi/4 \le t \le 2\pi$.

Then $||\mathbf{x}'(t)|| = \sqrt{\cos^2 t + (\cos t - t\sin t)^2 + 1}$.

Thus,

$$\int_{\mathbf{x}} f\, ds = \int_{\pi/4}^{2\pi} \frac{\sqrt{\cos^2 t + (\cos t - t\sin t)^2 + 1}}{t^2 \sin t \cos t}\, dt.$$

Not surprisingly, the path integral of a scalar field is intrinsic to the curve.

Theorem 6.4. Suppose that $\mathbf{x}\colon [a, b] \to \mathbb{R}^n$ is a C^1 curve and that $\mathbf{y}(t) = \mathbf{x}(h(t))$ is a reparameterization of $\mathbf{x}$ with some change of coordinates function $h\colon [c, d] \to [a, b]$. If $f\colon \mathbb{R}^n \to \mathbb{R}$ is a scalar field, continuous on an open set containing the image of $\mathbf{x}$, then

$$\int_{\mathbf{x}} f\, ds = \int_{\mathbf{y}} f\, ds.$$

Proof. By the chain rule,

$$\mathbf{y}'(h(t)) = \mathbf{x}'(h(t))h'(t).$$

Consequently,

$$||\mathbf{y}'(h(t))|| = ||\mathbf{x}'(h(t))|||h'(t)|.$$

Case 1: h is orientation preserving.

In this case $|h'(t)| = h'(t)$ and $h(c) = a$ and $h(d) = b$. Thus, using integration by substitution:

$$\begin{aligned} \int_{\mathbf{y}} f\,ds &= \\ \int_c^d f(\mathbf{y}(t))||\mathbf{y}'(t)||\,dt &= \\ \int_c^d f(\mathbf{x}(h(t))||\,||\mathbf{x}'(h(t))||h'(t)\,dt &= \\ \int_a^b f(\mathbf{x}(u))||\,||\mathbf{x}'(u)||\,dt &= \\ \int_{\mathbf{x}} f\,ds. \end{aligned}$$

Case 2: h is orientation preserving.

This case is almost the same as Case 1, except we use the fact that $|h'(t)| = -h'(t)$ and $h(c) = b$ and $h(d) = a$. ☐

As a final remark, we note that there is a major difference between the average value of a function on a curve averaged over *time* and the average value of a function on a curve averaged over *space*. For instance, if a swallow follows a path $\mathbf{x}(t)$ for $a \leq t \leq b$ through a room with a fire in it having temperature given by the function $f(\mathbf{x})$, the average temperature of the swallow is:

$$\frac{1}{b-a}\int_a^b f(\mathbf{x}(t))\,dt$$

but the average temperature of the path is

$$\frac{1}{\text{length of } \mathbf{x}}\int_{\mathbf{x}} f\,ds.$$

In general, these quantitites are not equal unless $\mathbf{x}$ is a parameterization by arc length. The first expression is an average over time which is an extrinsic quantity (it depends on the parameterization) and the second expression is an average over distance which is an intrinsic quantity (it depends only on the path, not on the parameterization.)

6.2. **Path Integrals of Vector Fields.**

Warm-up Question 1: Suppose that a constant force of f Newtons pushes a box d meters. How much work was done?

Answer: df Newton-meters of work was done, since if a force is constant and in the direction of motion, then the work done is equal to the magnitude of the force times the distance moved.

Warm-up Question 2: Suppose that a constant force of f Newtons is applied to a box and moves the box d meters. This time, however, the force is at an angle of θ degress with the direction of motion. How much work is done by the force?

Solution: Let $\mathbf{F}$ be the force and let $\mathbf{d}$ be the direction. The projection of $\mathbf{F}$ onto $\mathbf{d}$ tells us how much of the force is in the direction of motion. Some trigonometry tells us that it is $||\mathbf{F}|| \cos\theta$. Taking this times the distance travelled gives:

$$\text{Work done } = ||\mathbf{F}||||\mathbf{d}|| \cos\theta = \mathbf{F} \cdot \mathbf{d}.$$

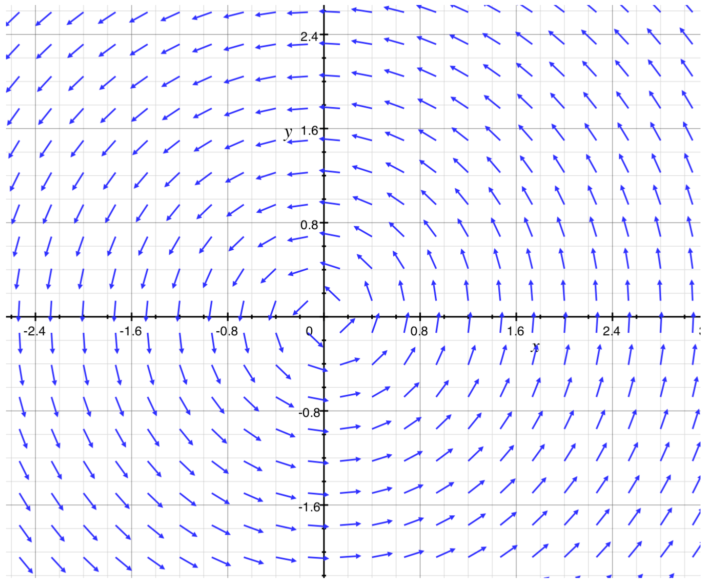
Warm-up Question 3: Suppose that at each point on a path $\mathbf{x}\colon [a, b] \to \mathbb{R}^n$ there is a different force $\mathbf{F}$. Assume that the path and the function $\mathbf{F}$ are C^1. How much work is done by the force to an object moving along the path?

Solution: Break the path into segments of equal length Δs and pretend that $\mathbf{F}$ is constant on each segment and that the image of $\mathbf{x}$ is a polygonal path with endpoints corresponding to the segment breaks. We would then add up $\mathbf{F} \cdot \mathbf{d}$ on each segment. This gives an approximation to the work done. If we want the exact value we take a limit. This suggests *defining* the work done to be:

$$\int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt.$$

A **vector field** on $\mathbb{R}^n$ is a function F such that for every $\mathbf{x} \in \mathbb{R}^n$, $F(\mathbf{x})$ is a vector in $T_{\mathbf{x}}$. Since $T_{\mathbf{x}}$ is simply a copy of $\mathbb{R}^n$ with origin at $\mathbf{x}$, we can think of F as the assignment of a vector $F(\mathbf{x})$ in $\mathbb{R}^n$ to each point in $\mathbb{R}^n$. Since we think of this vector as living in $T_{\mathbf{x}}$, we draw it as a vector in $\mathbb{R}^n$ with tail at $\mathbf{x}$.

Example 6.5. Here is a picture of the vector field $\mathbf{F}(x, y) = (-y, x)$. The arrows are not drawn with the correct lengths.



If $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field, and if $\mathbf{x}\colon [a,b] \to \mathbb{R}^n$ is a C^1 path, the **integral** of $\mathbf{F}$ along $\mathbf{x}$ is defined to be:

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt.$$

Example 6.6. Let $\mathbf{F}(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix}$. Let $\mathbf{x}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$. Find $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$.

Solution: Notice that $\mathbf{F}(\mathbf{x}(t)) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$ and that $\mathbf{x}'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$. Thus,

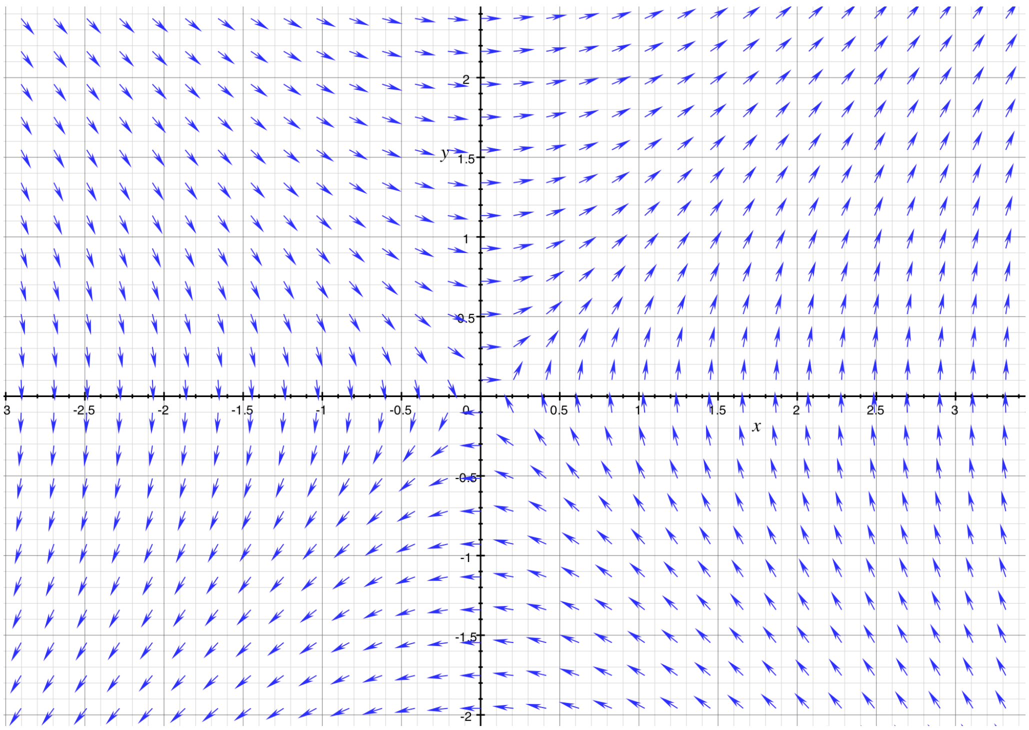
$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} = 2\pi.$$

Notice that here we have an example where the work done transporting an object in a closed loop is not necessarily zero.

Before discussing integration more, we look at additional examples of vector fields and some important associated concepts.

7. Vector Fields

Example 7.1. Here is the vector field $F(x, y) = (y, x)$. The arrows are not drawn with the right lengths.



A good way of thinking about a vector field is that it tells you the direction and speed of flow of water in a huge water system. To see this, suppose that we have an object in the stream at point (1,0) at time 0. Its position at time t is given by $\phi(t) = (x(t), y(t))$. If the vector field $F(x, y) = (F_1(x, y), F_2(x, y))$ describes the direction and speed of the object, then

$$\begin{aligned} x'(t) &= F_1(\phi(t)) \\ y'(t) &= F_2(\phi(t)) \end{aligned}$$

This a system of differential equations which we may or may not be able to solve. If ϕ exists, it is called a flow line for F.

Example 7.2. Find a flow line $\phi(t)$ for $\mathbf{F}(x, y) = (-y, x)$ passing through the point $(2, 0)$.

Solution: Suppose that $\phi(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. Then the equation $\phi'(t) = \mathbf{F}(\phi(t))$ becomes:

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -y(t) \\ x(t) \end{pmatrix}.$$

Thus we are looking for function x and y so that

$$\begin{aligned} x'(t) &= -y(t) \\ y'(t) &= x(t) \\ x(0) &= 2 \\ y(0) &= 0 \end{aligned}$$

The differential equations make us remember that sin and cos have derivatives related to each other in the way that we need.

Thus,

$$\phi(t) = \begin{pmatrix} 2\cos t \\ 2\sin t \end{pmatrix}$$

is the flow line we are looking for.

Example 7.3. Let $\mathbf{F}(x, y) = (y, x)$. Find flow lines through $(1, 1)$ and through $(1, 0)$.

Answer: Let $\phi(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ be a flow line. Then,

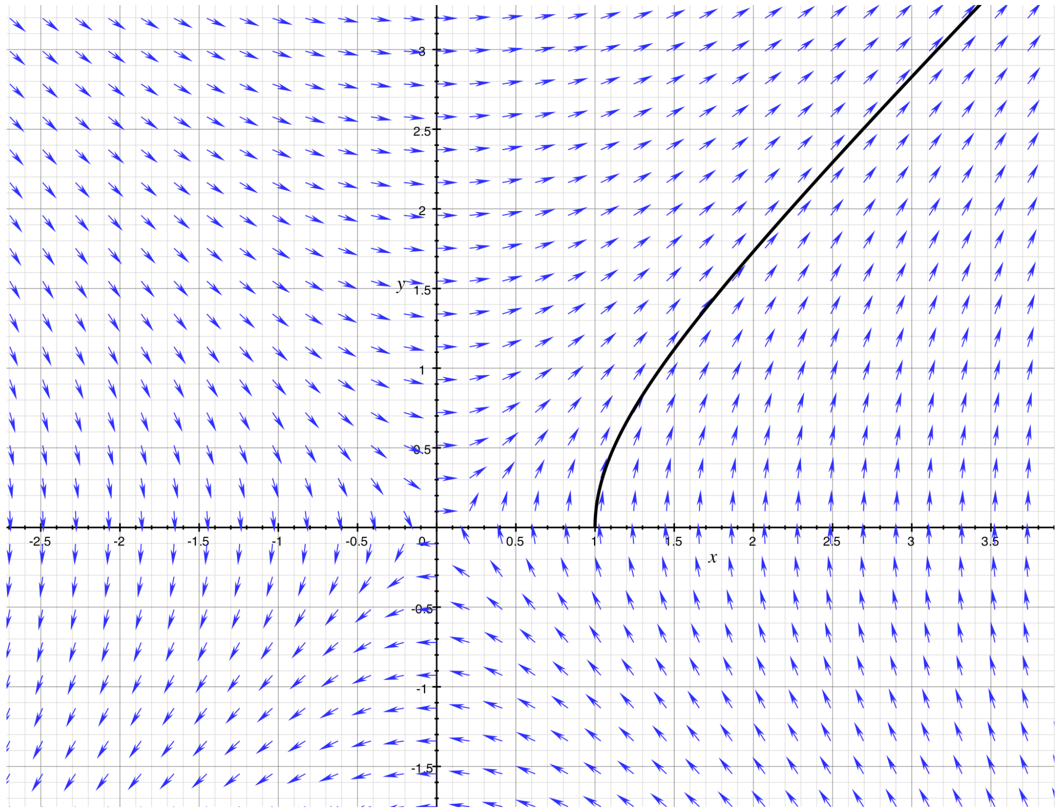
$$\begin{aligned} x'(t) &= y(t) \\ y'(t) &= x(t) \end{aligned}$$

As a first guess, we try $x(t) = e^t$ and $y(t) = e^t$. Sure enough, $\phi(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$ is a flow line for $\mathbf{F}$ passing through $(1, 1)$.

To find a flow line passing through $(1, 0)$ more ingenuity is required. Eventually, we might come up with:

$$\phi(t) = \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} = \begin{pmatrix} (e^t + e^{-t})/2 \\ (e^t - e^{-t})/2 \end{pmatrix}$$

The image of this second flow line in the vector field $\mathbf{F}$ is pictured below.



y
x

In this section, we study three ways of converting between scalar fields and vector fields: Gradient, Curl and Divergence. Here are the definitions. See below for examples.

Definition 8.1. If $f\colon \mathbb{R}^n \to \mathbb{R}$ is a differentiable scalar field, then the vector field ∇f is defined by:

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

The function f is called the **potential function** and the vector field ∇f is called **conservative vector field** or a **gradient vector field**. The operator ∇ that takes the scalar field f and convertes it to the vector field ∇f is called the **gradient**.

As explained below, the gradient field always points in the direction in which the potential function increases most rapidly.

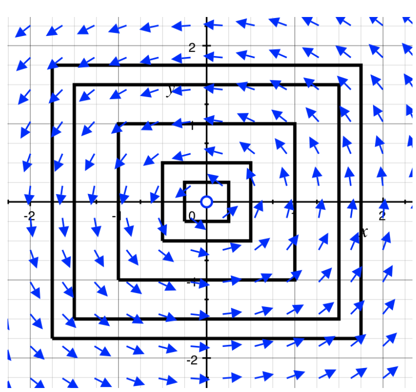
Definition 8.2. Suppose that $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a C^1 vector field defined on an open set containing a point $\mathbf{a}$. Suppose that $\mathbf{F} = \begin{pmatrix} M \\ N \end{pmatrix}$. The **scalar curl** of $\mathbf{F}$ is defined by either of the following formulas:

$$\text{scalar curl}\,\mathbf{F} = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial x}.$$

or

$$\text{scalar curl}\,\mathbf{F}(\mathbf{a}) = \lim_{n\to\infty} \frac{1}{\text{area}(C_n)} \int_{C_n} \mathbf{F} \cdot d\mathbf{s}$$

where C_n is a sequence of rectangles oriented counter-clockwise around $\mathbf{a}$ such that as $n \to \infty$, each point on C limits to $\mathbf{a}$. The area enclosed by C_n is denoted area(C_n). Here is a representative picture of a few of the C_n.



Below we show that these definitions are equivalent. The integral definition shows that scalar curl measures the "circulation per area" of a vector field at a point. In other words, it measures the amount of swirling at each point. Also, below we give the definition for "curl" of a vector field in $\mathbb{R}^3$.

Definition 8.3. Suppose that $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field defined on an open set containing the point $\mathbf{a}$. Suppose that $\mathbf{F} = \begin{pmatrix} M \\ N \end{pmatrix}$. The **divergence** of $\mathbf{F}$ is defined by the following formula:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

If $n = 2$, then divergence can also be defined by the formula:

$$\operatorname{div}(\mathbf{F})(\mathbf{a}) = \lim_{n\to\infty} \frac{1}{\operatorname{area}(C_n)} \int\limits_{C_n} \mathbf{F}\cdot\mathbf{n}\,ds.$$

Here, C_n is a sequence of rectangles enclosing a point $\mathbf{a}$ with the property that as $n \to \infty$ each point on the rectangles converges to $\mathbf{a}$. The vector $\mathbf{n}$ is the outward pointing unit normal vector to $\mathbf{C_n}$.

Below we show that if $n = 2$, these two definitions are equivalent. The integral definition shows that divergence measures the net amount of flow into or out of a point per area. We elaborate on this below.

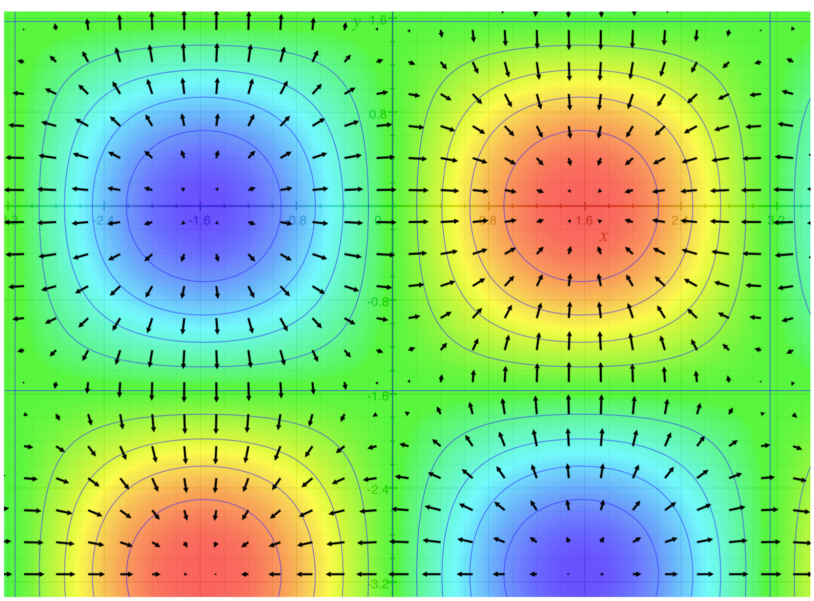
8.1. **Gradient.** As mentioned above, given a scalar field f, its gradient field ∇f is a certain vector field. The two basic questions are: Why do we care about gradient fields and how can we recognize them?

Here are some basic facts, that help answer those questions. We prove some of them below and more will be stated later.

- ∇f is perpendicular to contour lines for f. (Recall that a contour line is the set of points where f takes a constant value.)
- If f represents the temperature of air, the air will move along flow lines for $-\nabla f$ (since air moves from hotter to colder).
- The field ∇f does not have closed flow lines.
- The work done by ∇f on a particle moving along a closed path is zero.

8.1.1. *Examples.*

Example 8.4. Consider $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \sin x \cos y$. Then $\nabla f = (\cos x \cos y, -\sin x \sin y)$. Below is the vector field ∇f on top of the scalar field f. Contour lines have been drawn on the scalar field so that you can see how the vectors ∇f are perpendicular to the contour lines.



Example 8.5. The gravitational force exerted by a point mass at the origin on a point $\mathbf{x} \in \mathbb{R}^3$ is:

$$\mathbf{F}(\mathbf{x}) = \frac{-1}{||\mathbf{x}||^3}\mathbf{x}.$$

This is also the force exerted by a charged particle on a particle of opposite charge. This field is a gradient field with potential function:

$$f(\mathbf{x}) = \frac{1}{||\mathbf{x}||}.$$

Here's why:

Let $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ and $\mathbf{x} = (x, y, z)$. Finding the partials with respect to x, y, and z, we have:

$$\begin{array}{rcccl} \frac{\partial f}{\partial x}(x,y,z) & = & (-1/2)(2x)(x^2+y^2+z^2)^{-3/2} & = & \frac{-x}{||\mathbf{x}||^3} \\ \frac{\partial f}{\partial y}(x,y,z) & = & (-1/2)(2y)(x^2+y^2+z^2)^{-3/2} & = & \frac{-y}{||\mathbf{x}||^3} \\ \frac{\partial f}{\partial y}(x,y,z) & = & (-1/2)(2y)(x^2+y^2+z^2)^{-3/2} & = & \frac{-z}{||\mathbf{x}||^3} \end{array}$$

Thus,

$$\nabla f(x, y, z) = \frac{-1}{||x||^3}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{-\mathbf{x}}{||\mathbf{x}||^3}.$$

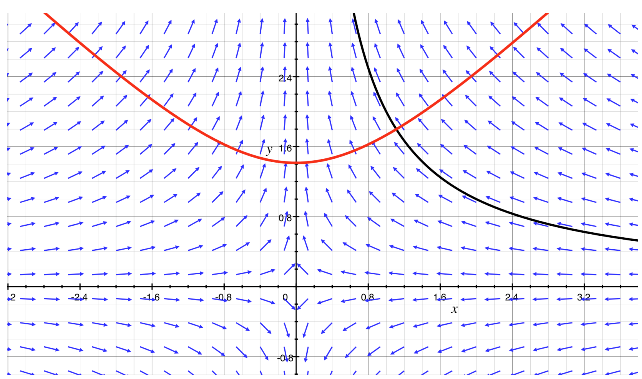
8.1.2. *Finding potential functions.*

Example 8.6. Find a potential function for $\mathbf{F}(x, y) = \begin{pmatrix} -x \\ y \end{pmatrix}$.

Answer: The function $f(x, y) = -\frac{1}{2}x^2 + \frac{1}{2}y^2$ is a potential function for $\mathbf{F}$ since $\nabla f = \mathbf{F}$. The hyperbolae

$$-\frac{1}{2}x^2 + \frac{1}{2}y^2 = c$$

are the equipotential lines for f. Notice in the figure below, that the equipotential line is perpendicular to a flow line. The flow line is black and the equipotential line is red.



The next example points to a general strategy for finding potential functions.

Example 8.7. Find a potential function for the vector field $\mathbf{F}(x, y) = \begin{pmatrix} 3x^2y + 2 \\ x^3 + 7 \end{pmatrix}$.

Answer: Let $M(x, y) = 3x^2y + 2$ and let $N(x, y) = x^3 + 7$. The antiderivatives of M with respect to x are:

$$\int M\,dx = x^3y + 2x + g(y)$$

where g is some function of y. The antiderivatives of N with respect to y are:

$$\int N\,dy = x^3y + 7y + h(x)$$

where h is some function of x. Putting these together we see that functions of the form:

$$f(x, y) = x^3y + 2x + 7y + C$$

for a constant C are potential functions of $\mathbf{F}$.

8.1.3. *Theoretical Results.*

Theorem 8.8 (Conservative Fields are perpendicular to equipotential lines)**.** Suppose that $\mathbf{F}$ is a conservative vector field with potential function f. Suppose that L is a smooth equipotential line for f and that ϕ is a flow line for $\mathbf{F}$ intersecting L. Then L and ϕ are perpendicular.

Proof. Suppose that ϕ and L intersect at a point $\mathbf{x_0}$ and that L has a unit tangent vector $\mathbf{v}$ at $\mathbf{x}_0$. Since f is constant along L, the directional derivative $\frac{\partial}{\partial \mathbf{v}} f(\mathbf{x_0})$ is equal to zero. By a standard result from Calculus II, $\frac{\partial}{\partial \mathbf{v}} f(\mathbf{x_0}) = \nabla f(\mathbf{x_0}) \cdot \mathbf{v}$. Since this is zero, $\nabla f(\mathbf{x_0}) = \mathbf{F}(\mathbf{x}_0)$ is perpendicular to L at $\mathbf{x_0}$. □

Another very useful fact is:

Theorem 8.9 (Conservative fields don't have closed flow lines)**.** Suppose that $\mathbf{F}$ is a gradient field and that ϕ is a flow line with $||\phi'(t)|| > 0$ for all t. Then ϕ does not close up on itself; in fact, for all t_1 and t_2 with $t_1 \neq t_2$, $\phi(t_1) \neq \phi(t_2)$.

Proof. Since $\mathbf{F}$ is a gradient field, there exists a potential function f for $\mathbf{F}$. Consider $g(t) = f(\phi(t))$. Then

$$g'(t) = Df(\phi(t))\phi'(t) = \nabla f(\phi(t)) \cdot \phi'(t)$$

Since $\mathbf{F} = \nabla f$ and since $\phi'(t) = \mathbf{F}(\phi(t))$, we have

$$g'(t) = \mathbf{F}(\phi(t)) \cdot \mathbf{F}(\phi(t)) = ||\mathbf{F}(\phi(t))||^2 = ||\phi'(t)||^2 > 0.$$

Thus, $g'(t) > 0$ for all t. In particular, $g(t) = f(\phi(t))$ is a strictly increasing function.

Suppose that there exist $t_1 \neq t_2$ such that $\phi(t_1) = \phi(t_2)$. Then $g(t_1) = g(t_2)$, but this contradicts the fact that g is strictly increasing. Hence, $\phi(t_1) \neq \phi(t_2)$ for all $t_1 \neq t_2$. □

Example 8.10. The vector field $\mathbf{F}(x, y) = (-y, x)$ has $\phi(t) = (\cos t, \sin t)$ as a flow line. Since $\phi(0) = \phi(2\pi)$, the vector field $\mathbf{F}$ is not a gradient field.

Example 8.11. The vector field $\mathbf{F}(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$ also has the unit circle as a closed flow line and so this vector field is not conservative either.

The most important theorem of this subsection is a version of the Fundamental Theorem of Calculus for conservative vector fields:

Theorem 8.12. Suppose that $\mathbf{F} = \nabla f$ is a C^1 conservative vector field on $\mathbb{R}^n$. Let $\mathbf{x}\colon [a,b] \to \mathbb{R}^n$ be a path, then

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{x}(b)) - f(\mathbf{x}(a)).$$

Notice that one implication of this theorem is that the path integral of a conservative vector fields depends only on the potential function and the endpoints of the path, but not on the path itself. This suggests the following definition (which applies to any vector field, not just conservative ones.)

Definition 8.13. A vector field $\mathbf{F}$ defined on a region $D \subset \mathbb{R}^n$ has **path independent line integrals** if whenever $\mathbf{x}\colon [a,b] \to D$ and $\mathbf{y}\colon [c,d] \to D$ are paths such that $\mathbf{x}(a) = \mathbf{y}(c)$ and $\mathbf{x}(b) = \mathbf{y}(d)$ we have

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s}.$$

In other words, line integrals depend only on the endpoints and direction of the path.

Theorem 8.12 has the immediate consequence:

Corollary 8.14. Conservative vector fields have path independent line integrals.

We now prove Theorem 8.12.

proof of Theorem 8.12. Recall that $\mathbf{F} = \nabla f$. By the chain rule:

$$\frac{d}{dt} f(\mathbf{x}(t)) = \nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t).$$

Hence,

$$\int_a^b \frac{d}{dt} f(\mathbf{x}(t))\, dt = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt.$$

We can evaluate the left hand side using the Fundamental Theorem of Calculus (version 2) to conclude that

$$f(\mathbf{x}(b)) - f(\mathbf{x}(a)) = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt.$$

The right hand side is just one piece of the definition of line integral and so we have:

$$f(\mathbf{x}(b)) - f(\mathbf{x}(a)) = \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$$

as desired. $\square$

Later, we will show that if a vector field has path independent line integrals, then it is conservative.

8.2. **Curl.** If C is an oriented closed curve, the **circulation** of a vector field $\mathbf{F}$ around C is defined to be $\int_C \mathbf{F} \cdot d\mathbf{s}$. The circulation is a measure of how much the vector field swirls around C. Notice that if the orientation (i.e. direction of traversing) is reversed then the circulation changes sign.

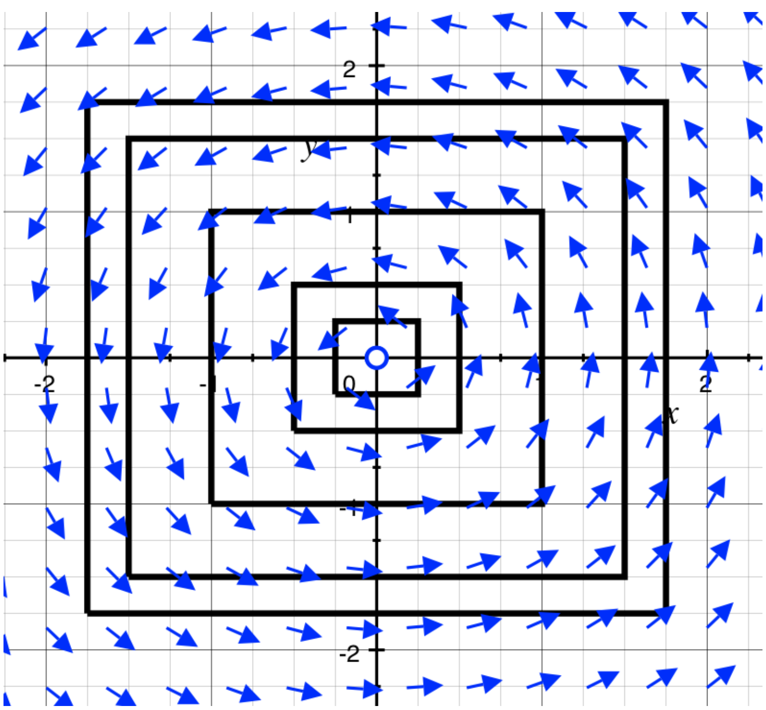
In the previous section, among other things, we showed that the circulation around any closed curve of a conservative vector field is 0. How can we use circulation to define a notion of "infinitessimal swirling" for vector fields? We integrate around a closed curve and then take a limit as it shrinks down to a point. Here are two preliminary examples:

Example 8.15. Let $\mathbf{F}(x, y) = \begin{pmatrix} -y \\ x \end{pmatrix}$. Let C_r be the circle $(r\cos t, r\sin t)$ for $0 \leq t \leq 2\pi$. Then:

$$\int_{C_r} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \begin{pmatrix} -r\sin t \\ r\cos t \end{pmatrix} \cdot \begin{pmatrix} -r\sin t \\ r\cos t \end{pmatrix} = 2\pi r^2.$$

As $r \to 0$, this produces 0. But notice that we have simply picked up twice the area enclosed by C_r. In fact if $\mathbf{F}$ is measure in, say, meters per second then the integrand as units $\mathrm{m}^2/\mathrm{sec}^2$ and so the integral has units $\mathrm{m}^2/\mathrm{sec}$. If we normalize by area (that is, divide the integral by the area enclosed by C_r then we obtain, in the limit, 2: this is a rate of rotation of $\mathbf{F}$ about the origin.

Example 8.16. Now let S_r be the square with corners at $(\pm r, \pm r)$ and let $\mathbf{F}(x, y) = \begin{pmatrix} -y \\ x \end{pmatrix}$ as before.



We can parameterize the sides of S_r by

$$\begin{array}{rlr} L_1(t) = & (t, -r) & -r \le t \le r \\ L_2(t) = & (r, t) & -r \le t \le r \\ L_3(t) = & (-t, r) & -r \le t \le r \\ L_4(t) = & (-r, -t) & -r \le t \le r. \end{array}$$

Notice that these have derivatives:

$$\begin{array}{rlr} L_1'(t) = & (1, 0) & -r \le t \le r \\ L_2'(t) = & (0, 1) & -r \le t \le r \\ L_3'(t) = & (-1, 0) & -r \le t \le r \\ L_4'(t) = & (0, -1) & -r \le t \le r. \end{array}$$

Then

$$\begin{array}{rcccl} \int_{L_1} \mathbf{F} \cdot d\mathbf{s} & = & \int_{-r}^{r} (r)\, dt & = & 2r^2. \\ \int_{L_2} \mathbf{F} \cdot d\mathbf{s} & = & \int_{-r}^{r} (r)\, dt & = & 2r^2 \\ \int_{L_3} \mathbf{F} \cdot d\mathbf{s} & = & \int_{-r}^{r} (r)\, dt & = & 2r^2 \\ \int_{L_4} \mathbf{F} \cdot d\mathbf{s} & = & \int_{-r}^{r} (r)\, dt & = & 2r^2 \end{array}$$

Thus,

$$\int_{S_r} \mathbf{F} \cdot d\mathbf{s} = 8r^2.$$

Dividing by the area enclosed by S_r, which is $4r^2$, we get 2, as before.

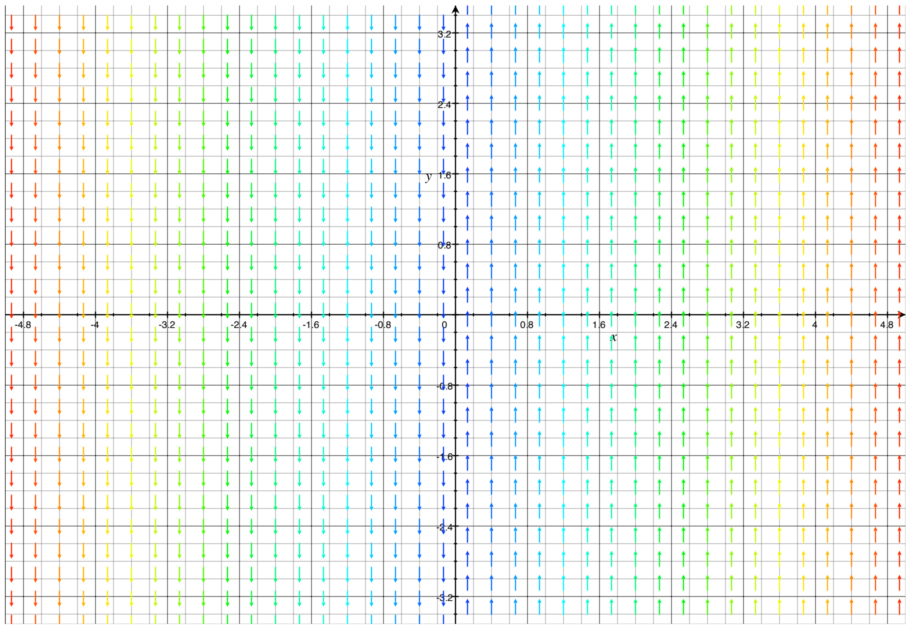
When we define infitessimal swirling, therefore, we need to define by the area enclosed by the curves shrinking down to the point where we are calculating the swirling. Also, it is technically easier to work with rectangles then with circles or other curves. After we prove Green's theorem, we'll be able to explain why it doesn't much depend what kind of curves we use.

Define the **scalar curl** of a two-dimensional vector field $\mathbf{F}$ and $\mathbf{a}$ by:

$$\text{scalar curl}\, \mathbf{F}(\vec{a}) = \lim_{S_r \to \mathbf{a}} \frac{1}{\text{area}(S_r)} \int_{S_r} \mathbf{F} \cdot d\mathbf{s}$$

where S_r is a sequence of rectangles with edges parallel to the x, y-plane and such that every point on S_r converges to $\mathbf{a}$.

Example 8.17. Let $\mathbf{F}(x, y) = \begin{pmatrix} 0 \\ x \end{pmatrix}$. Here is a picture of it:



Notice that the arrows get longer and longer as we move out the x axis. Let $S_n = [a, b] \times [c, d]$ be the rectangle with x-coordinates between a and b and y-coordinates between c and d. Think of S_n has being a rectangular raft located somewhere in the stream of water represented by $\mathbf{F}$. We begin by calculating the circulation of $\mathbf{F}$ around S_n. Since $\mathbf{F}$ is perpendicular to the bottom and top sides of S_n, we need only worry about the left and right sides. Parameterize these as:

$$\begin{array}{rcll} R(t) & = & (b, t) & c \leq t \leq d \\ L(t) & = & (a, t) & c \leq t \leq d \end{array}$$

Then

$$\int_{S_n} \mathbf{F} \cdot d\mathbf{s} = \int_R \mathbf{F} \cdot d\mathbf{s} - \int_L \mathbf{F} \cdot d\mathbf{s} = \int_c^d (b - a)\, dt = (b - a)(d - c).$$

Notice that the circulation of $\mathbf{F}$ around S_n is just the area enclosed by S_n. Choosing a sequence of rectangles converging to a point $\mathbf{a}$ we see that

$$\text{scalar curl}\, \mathbf{F}(\mathbf{a}) = 1$$

Since circulation divided by area is, in this case, 1 and the limit of 1 is 1.

We interpret the result as saying that no matter how small the raft, it will spin as it moves through the stream. Notice that the raft is moving in a straight line (because the flow lines of this vector field are straight lines), but is spinning as it does so.

We now set about showing that if $\mathbf{F}$ is a C^1 vector field, then its scalar curl is well -defined (meaning that the limit in the definition of scalar curl exists). It is most convenient to treat the horizontal and vertical components of $\mathbf{F} = \begin{pmatrix} M \\ N \end{pmatrix}$ separately. We begin with an analogue of the Mean Value Theorems from Calc 1.

Lemma 8.18. (Mean Value Theorem) Let $M\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function and let $C = [a,b]\times[c,d]$ be a rectangle such that M is defined on an open set containing C and its interior. Orient C counterclockwise. Then there exists $\mathbf{x_0} \in C$ so that $\frac{1}{(d-c)(b-a)} \int_C M\,dx = -\frac{\partial M}{\partial x}(\mathbf{x_0})$.

Proof. Since $x'(t) = 0$ we travel along the vertical sides of C, we need only pay attention to the horizontal sides. We parameterize the bottom side as (t,c) for $a \le t \le b$ and the top side as (t,d) for $a \le t \le b$. Notice that our parameterization of the top side is traversing it in the wrong direction. So we have:

$$\int_C M\,dx = \int_a^b M(t,c) - M(t,d)\,dt = -\int_a^b M(t,d) - M(t,c)\,dt.$$

By the Mean Value Theorem for Integrals, there exists $x_0 \in [a,b]$ so that

$$\frac{-1}{d-c}\cdot\frac{1}{(b-a)}\int_a^b M(t,d) - M(t,c)\,dt = (-1)\frac{M(x_0,d) - M(x_0,c)}{d-c}.$$

By the Mean Value Theorem for derivatives, there exists $y_0 \in (c,d)$ so that

$$(-1)\frac{M(x_0,d) - M(x_0,c)}{d-c} = (-1)\frac{\partial M}{\partial y}(x_0,y_0).$$

Letting $\mathbf{x} = (x_0,y_0)$, we have our result. ☐

Theorem 8.19. Let C_n be a sequence of rectangles converging to a point $\mathbf{a}$ as $n \to \infty$. Assume that $\mathbf{F} = (M,N)$ is a C^1 vector field defined on an open set containing all of the C_n and their interiors. Let area(C_n) be the area enclosed by C_n. Then,

$$\lim_{n\to\infty} \frac{1}{\operatorname{area}(C_n)} \int_{C_n} \mathbf{F}\cdot d\mathbf{s} = \frac{\partial N}{\partial x}(\mathbf{a}) - \frac{\partial M}{\partial y}(\mathbf{a}).$$

Proof. By Lemma 8.18, there exists $\mathbf{x}_n$ inside C_n, so that

$$\frac{1}{\operatorname{area}(C_n)} \int_{C_n} M\,dx = -\frac{\partial M}{\partial y}(\mathbf{x}_n).$$

An argument similar to that of the proof of Lemma 8.18, shows that there exists $\mathbf{y}_n$ inside C_n so that

$$\frac{1}{\operatorname{area}(C_n)} \int_{C_n} N\,dy = \frac{\partial N}{\partial x}(\mathbf{y}_n).$$

Alternatively, one can apply the result of Lemma 8.18 to the result of reflecting C_n and N across the line $y = x$. Since the partials of M and N are continuous, we have:

$$\lim_{n\to\infty} \frac{1}{\operatorname{area}(C_n)} \int_{C_n} M\,dx = \lim_{n\to\infty} -\frac{\partial M}{\partial y}(\mathbf{x}_n) = -\frac{\partial M}{\partial y}(\mathbf{a}),$$

and

$$\lim_{n\to\infty} \frac{1}{\operatorname{area}(C_n)} \int_{C_n} N\,dy = \lim_{n\to\infty} \frac{\partial N}{\partial x}(\mathbf{y}_n) = \frac{\partial N}{\partial x}(\mathbf{a}).$$

Consequently,

$$\lim_{n\to\infty} \frac{1}{\operatorname{area}(C_n)} \int_C \mathbf{F}\cdot d\mathbf{s} = \lim_{n\to\infty} \frac{1}{\operatorname{area}(C_n)} \Big(\int_{C_n} M\,dx + \int_{C_n} N\,dy.\Big)$$

Since limits are additive:

$$\lim_{n\to\infty} \frac{1}{\text{area}(C_n)} \int_C \mathbf{F}\cdot d\mathbf{s} = \left(\lim_{n\to\infty} \frac{1}{\text{area}(C_n)} \int_{C_n} M\,dx\right) + \left(\lim_{n\to\infty} \frac{1}{\text{area}(C_n)} \int_{C_n} N\,dy.\right.$$

By our previous observations this equals $\frac{\partial N}{\partial x}(\mathbf{a}) - \frac{\partial M}{\partial y}(\mathbf{a})$. $\square$

It is worth noting, that we (almost) always calculate scalar curl $\mathbf{F}$ with the derivative formula, but the integral formula tells us what the scalar curl is measuring.

Also note that since the integral of a conservative vector field around a closed curve is 0, we see immediately that the scalar curl of a conservative vector field at any point $\mathbf{a}$ is zero.

Can we define some sort of "curl" for vector fields in $\mathbb{R}^3$? We sure can: the curl of a vector field consists of a vector whose entries are the scalar curls in the yz, xz, and xy planes respectively.

Notice that if $\mathbf{F} = \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix}$, is actually 2-dimensional, then the $\mathbf{k}$ component of curl $\mathbf{F}(\mathbf{a})$ is the scalar curl of $\mathbf{F}$. Notice that, based on this definition, it is fair to say that curl measures the "infinitesimal rotation" of a vector field $\mathbf{F}$ about a point $\mathbf{a}$. Informally, if curl $\mathbf{F}(\mathbf{a}) \neq \mathbf{0}$, then a paddle-wheel dropped in the vector field at $\mathbf{F}$ will spin around an axis pointing in the direction of $curl\mathbf{F}(\mathbf{a})$. Although the definition of curl tells us what it is, it doesn't provide a good means of calculating it. Here's the formula:

Suppose that $\mathbf{F} = \begin{pmatrix} M \\ N \\ P \end{pmatrix}$. Then

$$\begin{aligned} \operatorname{curl}\mathbf{F}(\mathbf{a}) &= \nabla \times \mathbf{F}(\mathbf{a}) \\ &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial}{\partial y}P(\mathbf{a}) - \frac{\partial}{\partial z}N(\mathbf{a}) \\ \frac{\partial}{\partial z}M(\mathbf{a}) - \frac{\partial}{\partial x}P(\mathbf{a}) \\ \frac{\partial}{\partial x}N(\mathbf{a}) - \frac{\partial}{\partial y}M(\mathbf{a}) \end{pmatrix}. \end{aligned}$$

Example 8.20. Calculate the curl of $\mathbf{F}(x, y) = (-y, x)$.

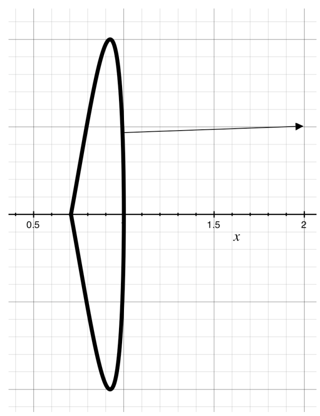
Solution:

$$
\begin{aligned}
\operatorname{curl} \mathbf{F}(x,y) &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial}{\partial y} 0 - \frac{\partial}{\partial z}(x) \\ \frac{\partial}{\partial z}(-y) - \frac{\partial}{\partial x}(0) \\ \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.
\end{aligned}
$$

8.3. **Divergence.** In the last section we saw two formulations of curl – one in terms of integrals and one in terms of partial derivatives. The formula in terms of integrals told us what curl means but the formula with derivatives tells us how to calculate it. In this section we'll do something similar for the "divergence" of a vector field. At present, we can only do this in 2-dimensions. After we introduce the notion of surface integral, we'll also be able to define divergence in 3-dimensions.

Before defining divergence, we need to define the concept of "outward pointing normal" to a curve.

Definition 8.21. Suppose that $\mathbf{x}\colon [a, b] \to \mathbb{R}^2$ is a simple closed curve (that is, it closes up and is one-to-one except at a and b.) Assume that $\mathbf{x}$ is (piecewise) C^1 and that for all t, $||x'(t)|| \neq 0$. A theorem from topology guarantees that the image of $\mathbf{x}$ separates $\mathbb{R}^2$ into two regions, one of which D is bounded. For $t \in [a, b]$, the **outward unit normal** to $\mathbf{x}$ is the unit vector $\mathbf{n}(t)$ that is perpendicular to $\mathbf{x}'(t)$ and points out of D. See the figure below. This may or may not be the same as the standard unit normal $\mathbf{N}(t)$.



Example 8.22. Let $\mathbf{x}(t) = \begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix}$. The outward pointing unit normal is $\mathbf{n}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$. If we reparameterize this circle, the outward unit normal $\mathbf{n}$ is the same at each point on the circle (although perhaps not at each moment in time).

To measure how much of $\mathbf{F}$ flows through a simple closed curve $\mathbf{x}$ at a particular point, we could look the dot product of $\mathbf{F}$ with $\mathbf{n}$ at that point. This suggests that the following definition will be useful.

Definition 8.23. Let C be a simple closed (piecewise) C^1 curve in $\mathbb{R}^2$ parameterized by $\mathbf{x}\colon [a, b] \to \mathbb{R}^2$. Let $\mathbf{n}(t)$ be the outward unit normal of $\mathbf{x}(t)$. If $\mathbf{F}$ is a C^1 vector field defined on an open set containing C, the **flux** of $\mathbf{F}$ through C is defined to be

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds$$

Notice that this is a line integral of a scalar field.

Example 8.24. Let $\mathbf{F}(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}$ and let C_r be a circle of radius r centered at the origin. Then the flux of $\mathbf{F}$ through C_r is:

$$\int_{C_r} \mathbf{F} \cdot \mathbf{n} \, ds.$$

To calculate this, parameterize C_r as $\mathbf{x}(t) = r(\cos t, \sin t)$ for $0 \leq t \leq 2\pi$ so that $\mathbf{n}(t) = (\cos t, \sin t)$. Recall that $||\mathbf{x}'(t)|| = r$.Then:

$$\int_{C_r} \mathbf{F} \cdot \mathbf{n} \, ds = \int_0^{2\pi} \begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix} \cdot \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \, r dt = 2\pi r^2.$$

Example 8.25. Let $\mathbf{F}(x, y) = (-y, x)$ and let C_r be the circle of radius r centered at the origin. Then the flux of $\mathbf{F}$ through C_r is 0.

If we want to measure how much a vector field $\mathbf{F}$ is spreading out from a point $\mathbf{a}$ or sucked into a point $\mathbf{a}$, we take the limit as $r \to 0^+$ of the flux through a rectangle containing $\mathbf{a}$ divide by the area enclosed.

Definition 8.26. Suppose that $\mathbf{F}$ is a C^1 vector field defined on an open subset of $\mathbb{R}^2$. If $\mathbf{a}$ is a point in the domain of $\mathbf{F}$, then the **divergence** of $\mathbf{F}$ at $\mathbf{a}$ is defined to be:

$$\operatorname{div} \mathbf{F}(vecta) = \lim_{n \to \infty} \frac{1}{\operatorname{area} C_n} \int_{C_n} \mathbf{F} \cdot \mathbf{n} \, ds.$$

where C_r is a rectangle with interior in the domain of $\mathbf{F}$ centered at $\mathbf{a}$ of radius r and $\mathbf{n}$ is its outward pointing unit normal. As $n \to \infty$, the rectangles C_n converge to $\mathbf{a}$.

Example 8.27. Based on our previous calculations we can deduce that the divergence of the vector field (x, y) at $\mathbf{0}$ is 2 and the divergence of $(-y, x)$ at $\mathbf{0}$ is 0.

Similar to the proof Theorem 8.19 we can show:

Theorem 8.28. Suppose that $\mathbf{F}(x, y) = \begin{pmatrix} M(x, y) \\ N(x, y) \end{pmatrix}$ is a C^1 vector field defined on an open subset of $\mathbb{R}^2$. Then

$$\operatorname{div} \mathbf{F}(x, y) = \frac{\partial}{\partial x} M(x, y) + \frac{\partial}{\partial y} N(x, y).$$

In fact, in all dimensions we could (and do) define the divergence of a vector field $\mathbf{F} = (F_1, \dots, F_n)$ to be

$$\operatorname{div} \mathbf{F}(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_i(\mathbf{x}).$$

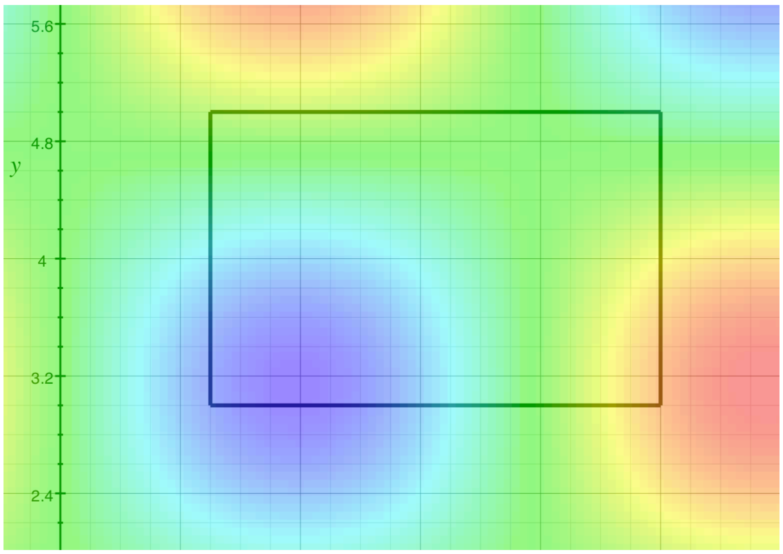
In the previous section we showed that the curl of a gradient field is $\mathbf{0}$. Now we show that the divergence of curl is 0.

Theorem 8.29. Suppose that $\mathbf{G}$ is a C^2 vector field defined on an open subset of $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\operatorname{div} \operatorname{curl} \mathbf{G} = 0$.

Theorem 8.30. This is an exercise that uses Theorem 8.28 and Theorem **??**.

9. Review: Double Integrals

9.1. **Integrating over rectangles.** Suppose that $R = [a, b] \times [c, d]$ is a rectangle in the xy-plane with corners at the points (a, c), (a, d), (b, c) and (b, d). Let $f\colon R \to \mathbb{R}$ is a scalar field. Here is a picture of the situation:



we define the double integral $\iint\limits_R f\,dA$ as follows.

Subdivide R into n rectangles $R_1, \ldots, R_n$ (each with sides parallel to the axes). Let ΔA_i denote the area of rectangle i. Choose a point $\mathbf{c_i}$ in rectangle R_i. Define the nth Riemann sum to be

$$S_n = \sum_{i=1}^{n} f(\mathbf{c_i})\Delta A_i$$

Notice that S_n is an approximation to the (signed) volume between the graph of f in $\mathbb{R}^3$ and the rectangle R in the xy-plane. Here is another interpretation of S_n: If all the rectangles are chosen to have the same area $\Delta A_i = \Delta A$, then $\Delta A = \text{Area}(R)/n$. The average value of f on the rectangle can be approximated by:

$$\begin{aligned}
\tfrac{1}{n}\textstyle\sum_{i=1}^{n} f(\mathbf{c_i}) &= \\
\tfrac{\Delta A}{\text{Area}(R)}\textstyle\sum_{i=1}^{n} f(\mathbf{c_i}) &= \\
\tfrac{1}{\text{Area}(R)}\textstyle\sum_{i=1}^{n} f(\mathbf{c_i})\Delta A &= \\
\tfrac{1}{\text{Area}(R)} S_n
\end{aligned}$$

We then define

$$\iint\limits_R f\,dA = \lim_{\max \Delta A_i \to 0} S_n$$

This integral represents the volume between the graph of f in $\mathbb{R}^3$ and the rectangle $R \subset \mathbb{R}^2$.

The average value of f could be defined to be

$$\frac{1}{\text{Area}(R)} \iint_R f\, dA.$$

Fubini's theorem is what we usually use to calculate double integrals. In many situations, Fubini's theorem tells us that a double integral can be rewritten as an iterated integral, that is as *two* Calc I integrals.

Theorem 9.1 (Fubini's theorem)**.** Suppose that $R = [a, b] \times [c, d]$ is a rectangle in $\mathbb{R}^2$ and that $f\colon R \to \mathbb{R}$ is a bounded function such that the discontinuities of f have zero area and every line parallel ot the coordinate axes meets the set of discontinuities in finitely many points. Then

$$\iint_R f\, dA = \int_a^b \int_c^d f(x, y)\, dydx = \int_c^d \int_a^b f(x, y)\, dxdy.$$

Example 9.2. Let $R = [0, 2\pi] \times [0, \pi]$ and let $f(x, y) = \sin x \cos y$. Find $\iint_R f\, dA$.

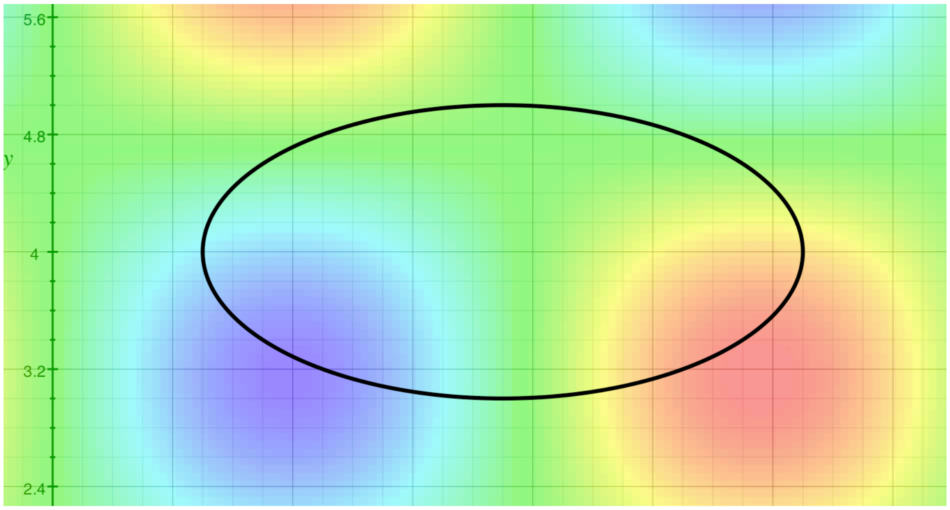
Solution: By Fubini's theorem:

$$\begin{aligned} \iint_R f\, dA. &= \int_0^{2\pi} \int_0^{\pi} \sin x \cos y\, dy\, dx \\ &= \int_0^{2\pi} \sin x \sin y\Big|_0^{\pi} dx \\ &= \int_0^{2\pi} 0\, dx \\ &= 0. \end{aligned}$$

9.2. **Integrating over non-rectangular regions.** There are two ways of defining an integral of a scalar field over a non-rectangular region. Both reduce the problem to integrating over rectangles. The first will suggest a good way of doing calculations but the second produces an integral that is defined in more situations. If both integrals exist, they give the same answer.

For our purposes, it will suffice to consider the situation when $D \subset \mathbb{R}^2$ is a closed and bounded region and when $f\colon D \to \mathbb{R}$ is a **continuous** function. We seek to define $\iint_D f\, dA$.

Here is a picture of a region D bounded by an ellipse and the scalar field $f(x, y) = \sin x \cos y$.

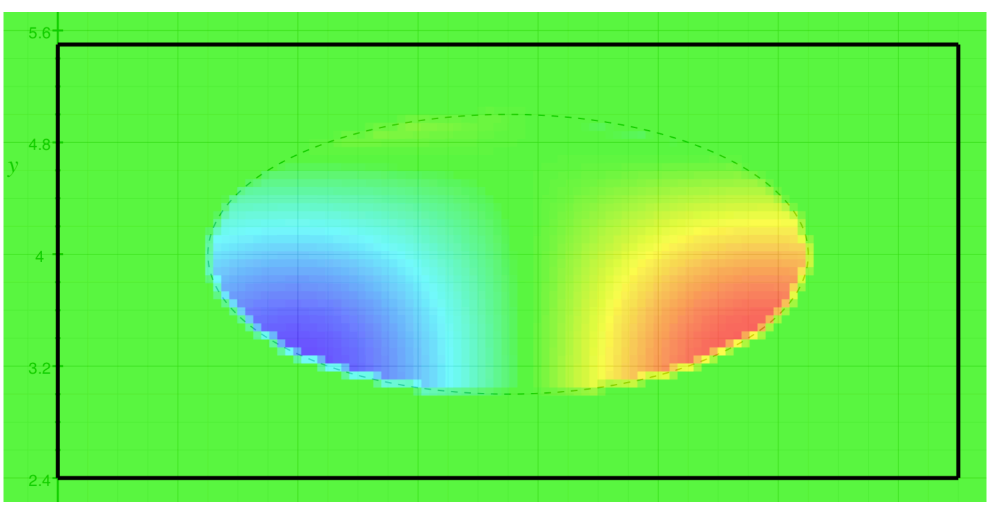


9.2.1. *The method of extension.* For this method we need to assume that the boundary of D (denoted by ∂D) has zero area and meets each vertical or horizontal line in only finitely many points. Since D is bounded, there is a rectangle R (with sides parallel to the axes) containing D in its interior.

We extend $f\colon D \to \mathbb{R}$ to a function $\widehat{f}\colon R \to \mathbb{R}$ defined by:

$$\widehat{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}.$$

Here is a picture of the scalar field $\widehat{f}$. The original ellipse is marked with a dashed line. Notice that outside of D, the scalar field is zero (i.e. green).



We now define:

$$\iint_D f\,dA = \iint_R \widehat{f}\,dA.$$

9.2.2. *The method of exhaustion.* For this method, we assume that D is an open set in $\mathbb{R}^2$. Subdivide the region D into m rectangles $R_1, \ldots, R_m$ so that $D \subset \bigcup_{i=1}^m C_i$. (That is the rectangles cover D.) Let ΔA be the maximum area of any rectangle. Choose the numbering so that rectangles $R_1, \ldots, R_n$ are completely contained inside D. Define:

$$\iint_D f\, dA = \lim_{\Delta A \to 0} \sum_{i=1}^n \iint_{R_i} f\, dA.$$

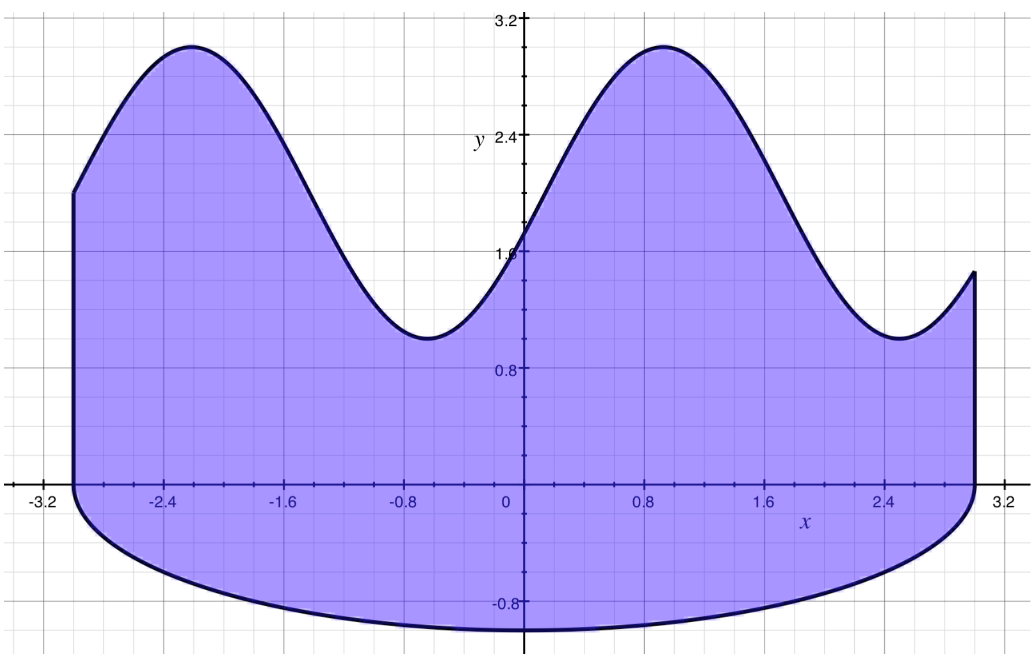
This integral is called the **improper integral** of f on D.

If D is a closed, bounded region in $\mathbb{R}^2$, we now have two possible definitions of $\iint_D f\, dA$. We could try to define it using the method of extension or define it using the method of exhaustion on the interior of the region. The improper integral will always exist, although the usual integral (defined using the method of extension) may not exist. If both exist, however, they are equal. See Theorem 15.4 of Munkres' *Analysis on Manifolds*.

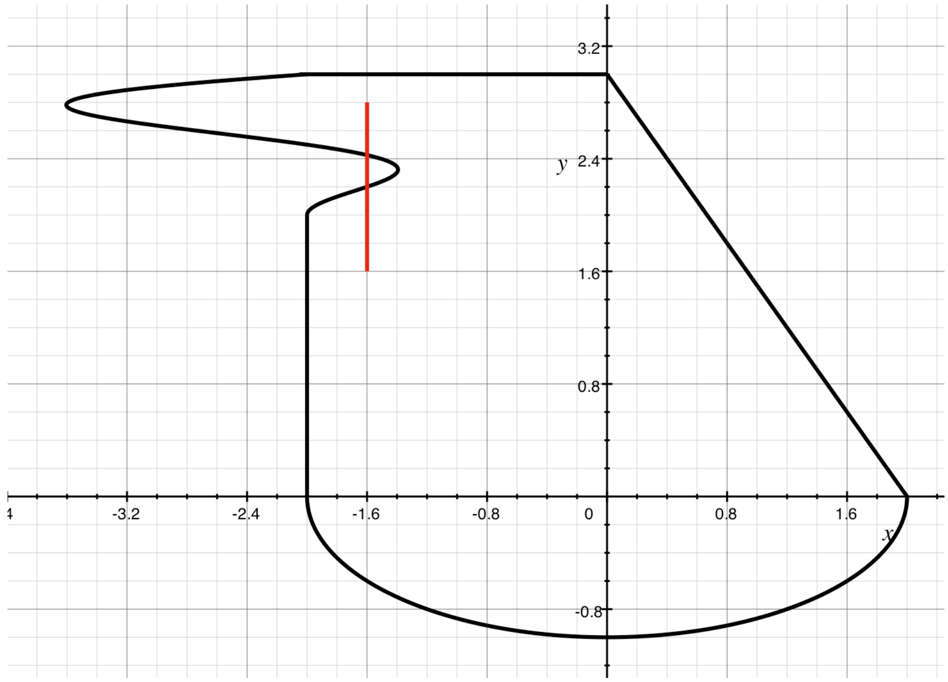
9.2.3. *Calculating integrals over non-rectangular regions.* The advantage of the definition of the integral $\iint_D f\, dA$ using the method of extension is that we can apply Fubini's theorem to the integral $\iint_R \widehat{f}\, dA$. Doing so provides the following methods of converting a double integral $\iint_D f\, dA$ over a non-rectangular region D into an iterated integral. To describe the method we first introduce some terminology.

Suppose that $D \subset \mathbb{R}^2$ is a closed, bounded region. We say that it is **vertically convex** (or **Type I**) if every vertical line segment having endpoints in D itself lies entirely in D.

Example 9.3. Here is an example of a vertically convex region:

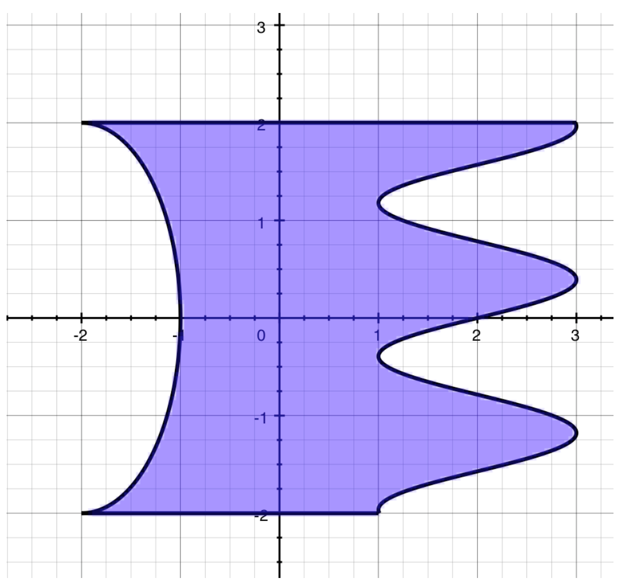


Example 9.4. The region bounded by the black curve is not vertically convex, since there exists a vertical line segment (in red) having both endpoints inside the region but not lying entirely in the region itself.



We say a closed, bounded region $D \subset \mathbb{R}^2$ is **horizontally convex** (or **Type II**) if every horizontal line segment with both endpoints in the region itself lies completely in the region.

Example 9.5. Here is an example of a horizontally convex region.



Finally, we say that a region is **Type III** if it is both of Type I and Type II. Regions bounded by circles and squares are examples of Type III regions.

Notice that if a region D is vertically convex, then it can be described as:

$$D = \left\{(x,y) \in \mathbb{R}^2 : \begin{array}{c} a \leq x \leq b \\ \gamma(x) \leq y \leq \delta(x) \end{array}\right\}$$

where γ and δ are functions of x.

Example 9.6. The region from Example 9.3 can be described as:

$$D = \left\{(x,y) \in \mathbb{R}^2 : \begin{array}{c} -3 \leq x \leq 3 \\ -\sqrt{1 - x^2/9} \leq y \leq \sin(2(x+3)) + 2 \end{array}\right\}$$

Notice that if a region D is horizontally convex, then it can be described as:

$$D = \left\{(x,y) \in \mathbb{R}^2 : \begin{array}{c} c \leq y \leq d \\ \alpha(y) \leq x \leq \alpha(y) \end{array}\right\}$$

where α and β are functions of y.

Example 9.7. The region from Example **??** can be described as:

$$D = \left\{(x,y) \in \mathbb{R}^2 : \begin{array}{c} -2 \leq y \leq 2 \\ \sqrt{1 - y^2/4} - 2 \leq x \leq \sin(4y) + 2 \end{array}\right\}$$

Now here's how to integrate a scalar field f over a non-rectangular region D **assuming that the integral defined by extension exists**.

- If D is vertically convex, write:

$$D = \left\{(x,y) \in \mathbb{R}^2 : \begin{array}{c} a \leq x \leq b \\ \gamma(x) \leq y \leq \delta(x) \end{array}\right\}$$

  Then

$$\iint\limits_D f\, dA = \int_a^b \int_{\gamma(x)}^{\delta(x)} f(x,y)\, dy\, dx.$$

- If D is horizontally convex, write:

$$D = \left\{(x,y) \in \mathbb{R}^2 : \begin{array}{c} c \leq y \leq d \\ \alpha(y) \leq x \leq \alpha(y) \end{array}\right\}$$

  Then

$$\iint\limits_D f\, dA = \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x,y)\, dx\, dy.$$

Example 9.8. Let D be the region from Example 9.3. Let $f(x,y) = xy^2$. Then:

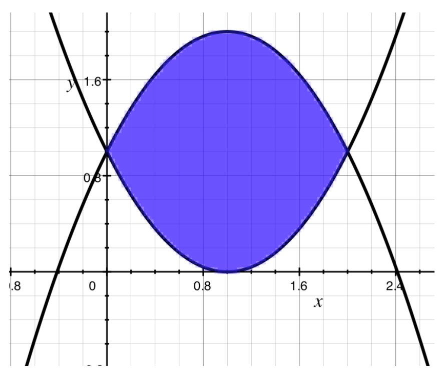
$$\iint_D f\,dA = \int_{-3}^{3} \int_{-\sqrt{1-x^2/9}}^{\sin(2(x+3)+2} xy^2\,dydx.$$

These integrals can easily be plugged into Mathematica or similar program.

Example 9.9. Let D be the region from Example **??**. Let $f(x,y) = \cos(xy)$. Then:

$$\iint_D f\,dA = \int_{-2}^{2} \int_{\sqrt{1-y^2/4}-2}^{\sin(4y)+2} \cos(xy)\,dA.$$

Example 9.10. Let D be the region between the graphs of $y = (x-1)^2$ and $y = -(x-1)^2+2$. Let $f(x,y) = xy$. Find $\iint_D f\,dA$.



Solution: Some easy algebra shows that the two curves intersect at the points $(0,1)$ and $(2,1)$. We are given the region as a Type I (vertically convex) region. So:

$$\iint_D f\,dA = \int_0^2 \int_{(x-1)^2}^{-(x-1)^2+2} xy\,dydx.$$

Sometimes there are other tricks we can use to find integrals.

Example 9.11. Let D be the square region with vertices at $(-1,0)$, $(1,0)$, $(0,1)$ and $(0,-1)$. Let $f(x,y) = x^2+y^2$. Find $\iint_D f\,dA$.

Solution: Notice that $f(x,y) = r^2$ where $r = ||(x,y)||$. Thus, if the plane is rotated about the origin by any angle $f(x,y)$ will remain unchanged. Rotate the plane so that D is a square centered at the origin and with sides parallel to the x and y axes. Call this new square D'. Since the value of f doesn't change after the rotation,

$$\iint_D f\,dA = \iint_{D'} f\,dA.$$

Both D and D' have sides of length $\sqrt{2}$ and so the corners of D' are at $(\pm\sqrt{2}, \pm\sqrt{2})$ and $(\pm\sqrt{2}, \mp\sqrt{2})$. Thus,

$$\iint\limits_{D} f\,dA = \iint\limits_{D'} f\,dA = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2}}^{\sqrt{2}} x^2 + y^2\,dxdy.$$

This last integral can be evaluated easily by hand:

$$\begin{aligned} \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2}}^{\sqrt{2}} x^2 + y^2\,dxdy &= \int_{-\sqrt{2}}^{\sqrt{2}} \tfrac{2}{3}(2^{3/2}) + 2\sqrt{2}y^2\,dy \\ &= \tfrac{4}{3}(2^{3/2})(2^{1/2}) + \tfrac{4\sqrt{2}}{3}(2^{3/2}) \\ &= \tfrac{32}{3} \end{aligned}$$

10.1. **0 and 1 dimensional integrals.** Let $I = [a, b]$ be an interval (oriented from a to b) If $F\colon I \to \mathbb{R}$ is a differentiable function, then you learn in one variable calculus that

$$\int_I \frac{d}{dt} F(t)\, dt = F(b) - F(a).$$

To generalize this theorem to higher dimensions we introduce some new terminology.

Terminology 1: If $p \in \mathbb{R}$ is a point, then say that p has "positive orientation" if we place an arrow on it pointing to the right. The point p has "negative orientation" if we put an arrow on it pointing to the left. If we have chosen an orientation for p, we say that p is oriented. If A is a finite subset of $\mathbb{R}$ and if each point in A has been given an orientation (not necessarily the same), we say that A is oriented.

Terminology 2: Suppose that $p \in \mathbb{R}$ is an oriented point and that $f\colon \mathbb{R} \to \mathbb{R}$ is a function. If p has positive orientation, define $\int_p f = f(p)$. If p has negative orientation, define $\int_p f = -f(p)$. If $A = \{p_1, \dots, p_n\}$ is a finite set of oriented points in $\mathbb{R}$, define $\int_A = \sum_{i=1}^n \int_{p_i} f$.

Terminology 3: Suppose that $a < b$ are real numbers. The interval $[a, b]$ is positively oriented and the interval $[ba]$ is negatively oriented. (Think of an arrow running from the small number a to the big number b. If the arrow points right, the interval is positively oriented; if it points left it is negatively oriented.) If I is an interval in $\mathbb{R}$ with endpoints $a < b$, then the "boundary" of I, denoted ∂I, is the set $\{a, b\}$. If I has positive orientation, we assign the points of ∂I the orientation with arrows pointing out of I. If I has negative orientation, we assign the points of ∂I, the orientations with arrows pointing into I. We say that ∂I has the orientation "induced" by the orientation from I.

Suppose that $I = [a, b]$ has positive orientation (i.e. $a < b$). Let $J = [b, a]$ be the same interval but with the opposite orientation. If $f\colon \mathbb{R} \to \mathbb{R}$ is integrable, then by definition

$$\int_I f = \int_a^b f(x)\, dx \qquad \text{and} \qquad \int_J f = \int_b^a f(x)\, dx = -\int_I f.$$

The fundamental theorem of calculus can then be stated as

Theorem 10.1 (Fundamental Theorem of Calculus)**.** Suppose that $F\colon \mathbb{R} \to \mathbb{R}$ is a C^1 function. Let $DF\colon \mathbb{R} \to \mathbb{R}$ be its derivative. Let $I \subset \mathbb{R}$ be an oriented interval and give ∂I the induced orientation. Then

$$\int_I DF = \int_{\partial I} F.$$

Notice that the FTC says that the 1-dimensional integal of a derivative is equal to a certain 0-dimensional integral (over the boundary) of the antiderivative.

10.2. **Green's theorem.**

Theorem 10.2 (Green's Theorem)**.** Suppose that $D \subset \mathbb{R}^2$ is closed and bounded and that ∂D is (piecewise) C^1. Orient ∂D so that D is always on the left. If $\mathbf{F}\colon D \to \mathbb{R}^2$ is a C^1

scalar field then:

$$\iint_D \text{scalarcurl}(\mathbf{F})\,dA = \int_{\partial D} \mathbf{F}\cdot d\mathbf{s}$$

Notice that this relates a 2-dimensional integral of a "derivative" to a 1-dimensional integral of an "anti-derivative".

Example 10.3. Let D_r be the disc of radius r in $\mathbb{R}^2$ centered at the origin. Let $C_r = \partial D_r$ oriented counterclockwise. Let $\mathbf{F}(xy) = \begin{pmatrix} -y \\ x \end{pmatrix}$. We calculate both sides of the equality in Green's theorem to verify that Green's theorem is true in this case.

The scalar curl of $\mathbf{F}$ is:

$$\text{scalarcurl}(\mathbf{F}(x,y)) = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 2.$$

Thus,

$$\iint_D \text{scalarcurl}(\mathbf{F}(x,y)\,dA = \iint_D 2\,dA = 2\,\text{Area}(D) = 2\pi r^2.$$

The curve C_r can be parameterized as $C_r(t) = \begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix}$ for $t \in [0, 2\pi]$. It is evident that $C_r'(t) = \begin{pmatrix} -r\sin t \\ r\cos t \end{pmatrix}$. Thus,

$$\int_{C_r} \mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(C_r(t))\cdot C_r'(t)\,dt = \int_0^{2\pi} r^2\,dt = 2\pi r^2$$

Notice that indeed:

$$\iint_D \text{scalarcurl}(\mathbf{F})\,dA = \int C_r \mathbf{F}\cdot d\mathbf{s}$$

Rather than using an orientation of ∂D that points in the direction of travel, we can also rephrase Green's theorem so as to use a normal orientation of ∂D.

Definition 10.4. Suppose that $D \subset \mathbb{R}^2$ is a closed, bounded region with ∂D piecewise C^1. At each C^1 point of ∂D, let $\mathbf{n}$ be the unit normal vector pointing out of D. If $\mathbf{x}(t) = (x(t), y(t))$ is a parameterization of ∂D with D always on the left, then the outward unit normal is $\mathbf{n}(t) = \frac{1}{||\mathbf{x}'(t)||}(y'(t), -x'(t))$.

Theorem 10.5 (Planar Divergence Theorem)**.** Suppose that $D \subset \mathbb{R}^2$ is a closed, bounded region with ∂D piecewise C^1. Let $\mathbf{n}$ be the outward pointing unit normal to ∂D. Then, if $\mathbf{F}$ is C^1:

$$\iint_D \operatorname{div}\mathbf{F}\,dA = \int_{\partial D} \mathbf{F}\cdot\mathbf{n}\,ds.$$

Notice that once again we have a 2-dimensional integral of a derivative equal to a 1-dimensional integral of an "antiderivative". (Obviously the terminology from Calc I doesn't match up exactly.) The right hand side of the equality above is called the "flux" of $\mathbf{F}$ across ∂D.

10.3. **Stokes' Theorem.** We don't have all the terminology available to us yet, but here's the statement of Stokes' theorem so that you can compare it to the other cases.

Theorem 10.6 (Stokes' Theorem)**.** Suppose that S is a (piecewise) C^1 orientable surface in $\mathbb{R}^3$ and that ∂S is also piecewise C^1. Give S an orientation and give ∂S the orientation induced from S. Let $\mathbf{F}$ be a C^1 vector field defined on an open set containing S. Then:

$$\iint S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int\limits_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

Once again we have a 2–dimensional integral of a derivative equal to a 1–dimensional integral of the anti-derivative. This theorem will eventually motivate us to define the notion of "surface integral". We will have to talk about parameterizing surfaces and orientations of surfaces, as well.

10.4. **The Divergence Theorem.** Once again, we don't have all the terminology available to us, but here is the statement of the Divergence Theorem (also known as Gauss' theorem).

Theorem 10.7 (The Divergence Theorem)**.** Suppose that $D \subset \mathbb{R}^3$ is a closed, bounded region with ∂D a piecewise C^1 surface. Give ∂D the normal orientation pointing out of D. Then

$$\iiint\limits_{D} \operatorname{div} \mathbf{F}\, dV = \iint\limits_{\partial D} \mathbf{F} \cdot d\mathbf{S}.$$

The integral on the left is the triple integral you encountered in Calc II. The integral on the right is a surface integral, which we have still to define. In any case, you can see that we have a 3-dimensional integral of a "derivative" equal to a 2–dimensional integral of the "anti-derivative".

10.5. **Generalized Stokes' Theorem.** The divergence theorem is as far as we'll be able to go in our class, but you may wonder about a version of FTC in dimensions greater than 3. There is such a thing, called "Generalized Stokes' Theorem" (or simply "Stokes' Theorem". Here, for the record is the statement. We won't define any of the unknown terms in this class.

Theorem 10.8 (Generalized Stokes' Theorem)**.** Suppose that M is an oriented, smooth n-dimensional manifold with smooth boundary ∂M having the induced orientation. (The boundary of M is an $(n-1)$ manifold.) Suppose that ω is an $(n-1)$ form on M having compact support then

$$\int\limits_{M} d\omega = \int\limits_{\partial M} \omega.$$

At the very least you can see an n-dimensional integral on the left and an $(n-1)$-dimensional integral on the right. The integrand on the left is a derivative of the integrand on the right. And that's all we'll say about that!

Theorem 11.1 (Green's Theorem)**.** Suppose that $D \subset \mathbb{R}^2$ is closed and bounded and that ∂D is (piecewise) C^1. Orient ∂D so that D is always on the left. If $\mathbf{F}\colon D \to \mathbb{R}^2$ is a C^1 scalar field then:

$$\iint\limits_D \text{scalarcurl}(\mathbf{F})\, dA = \int\limits_{\partial D} \mathbf{F} \cdot d\mathbf{s}$$

If we write $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$, then the conclusion of Green's theorem can be written as:

$$\int_{\partial D} M\, dx + N\, dy = \iint\limits_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

Before sketching a proof of Green's theorem, we'll look at some examples.

Example 11.2. For this example, let $D \subset \mathbb{R}^2$be the solid square with corners $(1,-1)$, $(1,1)$, $(-1,1)$, and $(-1,-1)$. We will need a parameterization of ∂D. Since ∂D is made up of 4 line segments, we can parameterize them as follows. For each of them $0 \leq t \leq 1$.

$$\begin{aligned} L_1(t) &= (1, 2t-1) \\ L_2(t) &= (1-2t, 1) \\ L_3(t) &= (-1, 1-2t) \\ L_4(t) &= (2t-1, -1) \end{aligned}$$

We will also need the derivatives:

$$\begin{aligned} L_1'(t) &= (0,2) \\ L_2'(t) &= (-2,0) \\ L_3'(t) &= (0,-2) \\ L_4'(t) &= (2,0) \end{aligned}$$

Example 1a: Let $\mathbf{F}(x,y) = (-x, y)$.

Example 1a.i: Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{s}$.

Answer: We have:

$$\begin{aligned} \int_{\partial D} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 \mathbf{F}(L_1(t)) \cdot L_1'(t)\, dt + \int_0^1 \mathbf{F}(L_2(t)) \cdot L_2'(t)\, dt + \\ &\quad \int_0^1 \mathbf{F}(L_3(t)) \cdot L_3'(t)\, dt + \int_0^1 \mathbf{F}(L_4(t)) \cdot L_4'(t)\, dt \\ &= \int_0^1 \begin{pmatrix} -1 \\ 2t-1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 2t-1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1-2t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 1-2t \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} dt \\ &= \int_0^1 2(2t-1) + (-2)(2t-1) + (-2)(1-2t) + 2(1-2t)\, dt \\ &= 0 \end{aligned}$$

Example 1a.ii: Compute $\iint\limits_D (\text{curl}\, \mathbf{F}) \cdot \mathbf{k}\, dA$.

Answer: We have

$$\text{curl}\, \mathbf{F} \cdot \mathbf{k} = \frac{\partial(y)}{\partial x} - \frac{\partial(-x)}{\partial y} = 0.$$

Thus, $\iint\limits_D \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA = \iint\limits_D 0\, dA = 0$. Notice that this matches the answer from Example 1a.i, as predicted by Green's theorem.

Example 1b: Let $\mathbf{F}(x, y) = (-y, x)$.

Example 1b.i Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{s}$.

Answer: We have:

$$\begin{aligned}
\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 \mathbf{F}(L_1(t)) \cdot L_1'(t)\, dt + \int_0^1 \mathbf{F}(L_2(t)) \cdot L_2'(t)\, dt + \\
&\quad \int_0^1 \mathbf{F}(L_3(t)) \cdot L_3'(t)\, dt + \int_0^1 \mathbf{F}(L_4(t)) \cdot L_4'(t)\, dt \\
&= \int_0^1 \begin{pmatrix} 1-2t \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1-2t \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2t-1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2t-1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} dt \\
&= \int_0^1 2 + 2 + 2 + 2\, dt \\
&= 8
\end{aligned}$$

Example 1b.ii Compute $\iint\limits_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dA$.

In this case, $\operatorname{curl} \mathbf{F} \cdot \mathbf{k} = 2$. Thus,

$$\iint\limits_D \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA = \int_{-1}^1 \int_{-1}^1 2\, dA = 8.$$

Notice that this is the same as in Example 1b.i as predicted by Green's theorem.

Example 11.3. Let $\mathbf{F}(x, y) = (\sin x, \ln(1 + y^2))$. Let C be a simple closed curve which is made up of 24 line segments in a star shape. Compute $\int_C \mathbf{F}\, d\mathbf{s}$.

Answer: Let D be the region bounded by C. Notice that $\operatorname{curl} \mathbf{F} = \mathbf{0}$, so $\iint\limits_D \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA = 0$. By Green's theorem, this is also the answer to the requested integral.

Example 11.4. Let $\phi(t) = \begin{pmatrix} \cos t \sin(3t) \\ \sin t \cos(3t) \end{pmatrix}$ for $0 \le t \le \pi/2$. Find the area of the region D enclosed by ϕ.

Answer: Notice that ϕ travels clock-wise around D, we need it to go counter-clockwise to use Green's theorem. Changing the direction that ϕ travels, changes the sign of a path integral of a vector field. Thus, by Green's theorem, the area of D is given by

$$\iint_D 1\, dA = -\int_\phi \mathbf{F} \cdot d\mathbf{s},$$

where $\mathbf{F}$ is a vector field having the property that $\operatorname{curl}\mathbf{F} = (0,0,1)$. The vector field: $\mathbf{F}(x,y) = \frac{1}{2}(-y,x)$ has that property. Thus,

$$\begin{aligned}
\iint_D 1\,dA &= -\int_\phi \mathbf{F}\cdot d\mathbf{s} \\
&= -(1/2)\int_0^{\pi/2}(-\sin t\cos 3t, \cos t\sin 3t)\cdot\phi'(t)\,dt \\
&= -(1/2)\int_0^{\pi/2}\cos 3t\sin 3t - 3\sin t\cos t\,dt \\
&= -(1/2)\int_0^{\pi/2}\sin(6t)/2 - 3\sin(2t)/2\,dt \\
&= -(1/2)(1/6 - 3/2) \\
&= 2/3.
\end{aligned}$$

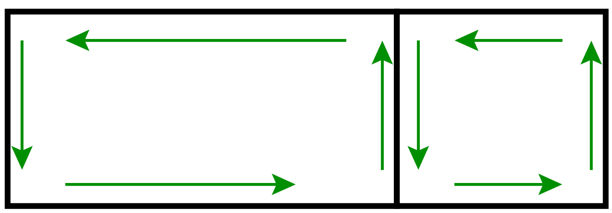
The proof of Green's theorem relies heavily on the following fact:

Lemma 12.1 (Cancelling Edges)**.** If S_1 and S_2 are two regions sharing an edge e (as in the figure below) and each oriented counterclockwise and if $\mathbf{F}$ is a C^1 vector field defined over both S_1 and S_2, then

$$\int_{\partial S_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\partial S_2} \mathbf{F} \cdot d\mathbf{s} = \int_{\partial(S_1 \cup S_2)} \mathbf{F} \cdot d\mathbf{s}.$$

That is, the integrals over the edge e cancel.

Proof. This follows immediately from the fact that the counterclockwise orientations on the boundaries of S_1 and S_2 go opposite ways around the edge e. □



Also recall that we proved the following Mean Value Theorem for Line Integrals

Theorem 12.2. Suppose that R is a rectangle with ∂R oriented counterclockwise and that $\mathbf{F}$ is a C^1 vector field defined over R. Assume that $\mathbf{F}$ is of the form $\begin{pmatrix} M \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ N \end{pmatrix}$. Then there exists $\mathbf{a} \in R$ so that

$$\int_{\partial R} \mathbf{F} \cdot d\mathbf{s} = \text{scalar curl}\, \mathbf{F}(\mathbf{a})\, \text{area}(R).$$

We can now sketch a proof of Green's theorem. At the end we comment on why our proof is quite rigorous enough.

Sketch Proof of Green's Theorem. Recall that S is a surface contained in $\mathbb{R}^2$ and that $\mathbf{F}$ is a C^1 vector field defined on S. Subdivide S using a grid of horizontal and vertical straight lines. Let $R_1, \dots, R_n$ be the pieces. If the grid is fine enough, most of the R_i are rectangles. Unless ∂S is made up entirely of horizontal and vertical line segments, not all the R_i are rectangles though. The only ones that aren't have pieces of ∂S in their boundary. By the Cancelling edges lemma, we have:

$$\sum \int_{\partial R_i} \mathbf{F} \cdot d\mathbf{s} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

By the Mean Value Theorem for Line Integrals, there exists $\mathbf{a}_i \in R_i$ (when R_i is a rectangle) such that

$$\int_{\partial R_i} \mathbf{F} \cdot d\mathbf{s} = \text{scalar curl}\, \mathbf{F}(\mathbf{a}_i)\, \text{area}(R_i).$$

Adding these up over all R_i we get:

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \sum \int_{\partial R_i} \mathbf{F} \cdot d\mathbf{s} = \sum \text{scalar curl}\, \mathbf{F}(\mathbf{a}_i)\, \text{area}(R_i).$$

Taking the limit of the left and right sides as the grid gets finer and finer gives us:

$$\lim \int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \lim \sum \text{scalar curl}\, \mathbf{F}(\mathbf{a}_i)\, \text{area}(R_i).$$

But the term on the left doesn't change as the limit is taken and the term on the right is the limit of Riemann sums and so:

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_S \text{scalar curl}\, \mathbf{F}\, dA.$$

See Figure 3 □

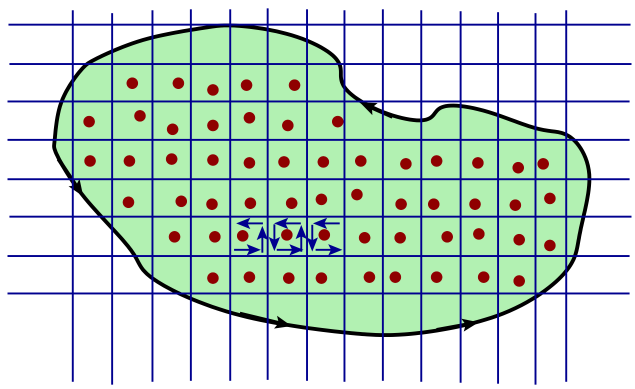


FIGURE 3. The green surface S is broken up into subrectangles. The orientations on the boundaries of the subrectangles cause lots of cancellation when their line integrals are computed, leaving only $\int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$. Each integral around the boundary of one of subrectangles is equal to the scalar curl of the vector field at a particular point (marked in red) times the area of the subrectangle.

What makes this "proof" of Green's theorem incomplete is the fact that we overlooked the fact that not all the R_i are rectangles. There are ways of dealing with this, but they are beyond the scope of this course.

13. Applications of Green's Theorem

13.1. **Finding Areas.** Green's theorem says that (under certain hypotheses)

$$\iint\limits_D \text{scalar curl}\, \mathbf{F}\, dA = \int\limits_{\partial D} \mathbf{F} \cdot d\mathbf{s}$$

We recall that the area of a region D is equal to $\iint\limits_D 1\, dA$. Thus, if we can find a vector field $\mathbf{F}$ with scalar curl $\mathbf{F} = 1$ we can use Green's theorem to compute the areas of regions bounded by parameterized curves. There are three vector fields having scalar curl 0 that are most commonly used for computing area. They are $\begin{pmatrix} 0 \\ x \end{pmatrix}$, $\begin{pmatrix} -y \\ 0 \end{pmatrix}$, and $\begin{pmatrix} -y/2 \\ x/2 \end{pmatrix}$. Which one we use depends on the situation or personal preference.

Example 13.1. Compute the area enclosed by a circle of radius R using Green's theorem.

Solution: Let D be the disc of radius R centered at the origin and parameterize ∂D as $\mathbf{x}(t) = \begin{pmatrix} R\cos t \\ R\sin t \end{pmatrix}$ for $t \in [0, 2\pi]$. We have chosen our parameterization so that D is on the left, as we needed to apply Green's theorem. Let $\mathbf{F}(x, y) = \begin{pmatrix} 0 \\ x \end{pmatrix}$. Then Green's Theorem says:

$$\begin{aligned}
\iint_D 1\, dA &= \iint_D \text{scalar curl}\, \mathbf{F}\, dA \\
&= \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} \\
&= \int_0^{2\pi} \begin{pmatrix} 0 \\ R\cos t \end{pmatrix} \cdot \begin{pmatrix} -R\sin t \\ R\cos t \end{pmatrix} dt \\
&= \int_0^{2\pi} R^2 \cos^2 t\, dt \\
&= \int_0^{2\pi} R^2 (\cos 2t + 1)/2\, dt \\
&= \pi R^2
\end{aligned}$$

You might try this example using a different choice of $\mathbf{F}$ to see if it is easier.

Example 13.2. Compute the area enclosed by the ellipse $\mathbf{x}(t) = \begin{pmatrix} 2\cos t \\ \sin t \end{pmatrix}$ with $t \in [0, 2\pi]$.

Solution: Let $\mathbf{F}(x, y) = \begin{pmatrix} -y/2 \\ x/2 \end{pmatrix}$. As in the previous example, by Green's theorem, the area enclosed by the ellipse is equal to

$$\begin{aligned}
\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} \begin{pmatrix} -(\sin t)/2 \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} -2\sin t \\ \cos t \end{pmatrix} dt \\
&= \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt \\
&= 2\pi
\end{aligned}$$

Example 13.3. Let $\phi(t) = \begin{pmatrix} \cos t \sin(3t) \\ \sin t \cos(3t) \end{pmatrix}$ for $0 \le t \le \pi/2$. Find the area of the region D enclosed by ϕ.

Answer: Notice that ϕ travels clock-wise around D, we need it to go counter-clockwise to use Green's theorem. Changing the direction that ϕ travels, changes the sign of a path integral of a vector field. Thus, by Green's theorem, the area of D is given by

$$\iint_D 1\, dA = -\int_\phi \mathbf{F} \cdot d\mathbf{s},$$

where $\mathbf{F}$ is a vector field having the property that scalar curl $\mathbf{F} = 1$. Let $\mathbf{F}(x,y) = \frac{1}{2}(-y,x)$. Then,

$$\begin{aligned}
\iint_D 1\,dA &= -\int_\phi \mathbf{F}\cdot d\mathbf{s} \\
&= -(1/2)\int_0^{\pi/2}(-\sin t\cos 3t, \cos t\sin 3t)\cdot \phi'(t)\,dt \\
&= -(1/2)\int_0^{\pi/2}\cos 3t\sin 3t - 3\sin t\cos t\,dt \\
&= -(1/2)\int_0^{\pi/2}\sin(6t)/2 - 3\sin(2t)/2\,dt \\
&= -(1/2)(1/6 - 3/2) \\
&= 2/3.
\end{aligned}$$

13.2. **Conservative Vector Fields.** In previous sections we've studied conservative vector fields and have deduced the following:

- If any of the following hold, the vector field $\mathbf{F}$ is **not** conservative:
  - $\mathbf{F}$ has a closed up flow line (assuming $||\mathbf{F}|| > 0$.)
  - $\operatorname{curl} \mathbf{F} = \mathbf{0}$.
  - There is a closed curve C such that $\int_C \mathbf{F} \cdot d\mathbf{s} \neq 0$
  - There are two paths $\mathbf{x}$ and $\mathbf{y}$ both connecting point $\mathbf{a}$ to point $\mathbf{b}$ such that $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} \neq \int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s}$.
- If any of the following hold, the vector field $\mathbf{F}$ **is** a conservative field:
  - There exists a scalar field f (with the same domain as $\mathbf{F}$) such that $\mathbf{F} = \nabla f$.
  - $\mathbf{F}$ has path independent line integrals
  - For all closed curves C, the integral $\int_C \mathbf{F} \cdot d\mathbf{s}$ is zero.

Thus while we have practical ways of determining that a vector field **F is not** conservative, the only even semi-practical way we have of determining that **F is** conservative is to produce a potential function for $\mathbf{F}$. Usually, this involves solving a system of partial differential equations, and usually that is extremely difficult. Still, if we can find a potential function that's very useful. The next theorem applies Green's theorem to produce a simple criterion for a vector field to be conservative. Before stating the theorem, we make a definition:

Definition 13.4. An open subset X of $\mathbb{R}^n$ is called **connected** if for all points $\mathbf{a}$ and $\mathbf{b}$ in X, there is a path in X joining $\mathbf{a}$ to $\mathbf{b}$. (That is, X is "one piece"[1])We say that X has **all loops contractible** if for each closed curve C in X, the curve C can be shrunk to a point in X all the while remaining in X. If X is connected and has all loops contractible, we say that it is **simply connected**.

Example 13.5. In $\mathbb{R}^2$, a connected open subset X is simply connected if and only if it "has no holes". The set $\mathbb{R}^3 - \{\mathbf{0}\}$ is simply connected, but $\mathbb{R}^3 - \{(0, 0, z) : z \in \mathbb{R}\}$ is not simply connected.

Theorem 13.6. Suppose that X is a simply connected open subset of $\mathbb{R}^2$ or $\mathbb{R}^3$. If $\mathbf{F}$ is a C^1 vector field on X such that $\operatorname{curl} \mathbf{F} = \mathbf{0}$ then $\mathbf{F}$ is a conservative vector field.

Proof Sketch. We (partially) prove this for $X \subset \mathbb{R}^2$ using Green's theorem. To prove it for $X \subset \mathbb{R}^3$, you should use Stokes' theorem which we will discuss later. (You would also need a replacement for the Jordan Curve Theorem, so it's rather challenging to do this in 3D.)

The exact statement that we will prove is that: "If $X \subset \mathbb{R}^2$ is open and simply connected, and if $\mathbf{F}$ is a C^1 vector field on X with $\operatorname{curl} \mathbf{F} = \mathbf{0}$, then if C is a simple closed curve $\int_C \mathbf{F} \cdot d\mathbf{s} = \mathbf{0}$."

If we could prove this statement for any closed curve C (not just simple closed curve) then by the observations before the theorem we would know that $\mathbf{F}$ was conservative, as desired. This gap between what we need to prove and what we will prove is one reason why I say this is just a proof sketch.

[1] Technically, being "one piece" is called "connected" while having the property that any two points can be joined by a path is called "path connected". There are spaces where these concepts differ. In our context, however, they are the same.

We assume by hypothesis that $\operatorname{curl} \mathbf{F} = \mathbf{0}$ and that $X \subset \mathbb{R}^2$ is simply connected. Let $C \subset X$ be a simple closed curve. A theorem from topology (the Jordan Curve Theorem) says that in $\mathbb{R}^2$, the curve C is the boundary of a compact region D. Since X is simply connected, the region D must be a subset of X. Thus, $\mathbf{F}$ is defined on D. By Green's theorem, $\int_C \mathbf{F} \cdot d\mathbf{s} = \iint\limits_D \text{scalar curl}\, \mathbf{F}\, dA = 0$. □

We put our previous results together to obtain:

Theorem 13.7 (Poincaré)**.** Let $X \subset \mathbb{R}^2$ (or $\mathbb{R}^3$) be a simply connected, open domain. Let $\mathbf{F}$ be a C^1 vector field defined on X. Then, the following are equivalent:

(1) There exists a potential function f with domain X such that $\mathbf{F} = \nabla f$.
(2) $\mathbf{F}$ has path independent line integrals on X
(3) The integral of $\mathbf{F}$ around any closed curve in X is zero.
(4) $\operatorname{curl} \mathbf{F} = \mathbf{0}$.

13.3. **Planar Divergence Theorem.** Recall that if C is a smooth simple closed curve in $\mathbb{R}^2$, the vector $\mathbf{n}$ at $(x, y) \in C$ is the outward pointing unit normal vector. If $\mathbf{F}$ is a C^1 vector field, $\int_C \mathbf{F} \cdot \mathbf{n} \cdot d\mathbf{s}$ is called the **flux** of $\mathbf{F}$ through C. The planar divergence theorem relates the flux of $\mathbf{F}$ through C to the total amount of divergence of $\mathbf{F}$ inside the region bounded by C.

Theorem 13.8 (Planar Divergence Theorem)**.** Suppose that $D \subset \mathbb{R}^2$ is a closed, bounded region with ∂D piecewise C^1. Let $\mathbf{n}$ be the outward pointing unit normal to ∂D. Then, if $\mathbf{F}$ is C^1 on D:

$$\iint\limits_D \operatorname{div} \mathbf{F} \, dA = \int_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds.$$

Proof. Let $\mathbf{G}$ be the vector field obtained from $\mathbf{F}$ by rotating each vector 90° counterclockwise. If $\mathbf{F} = (M, N)$ then $\mathbf{G} = (-N, M)$. Rotating $\mathbf{n}$ by 90° counterclockwise creates the unit tangent vector $\mathbf{T}$ to C oriented so that D is on the left as we traverse C in the direction of $\mathbf{T}$. Since the magnitude of $\mathbf{G}$ at each point equals the magnitude of $\mathbf{F}$ at that point and since the angle between $\mathbf{G}$ and $\mathbf{T}$ is the same as the angle between $\mathbf{F}$ and $\mathbf{n}$, we have $\mathbf{G} \cdot \mathbf{T} = \mathbf{F} \cdot \mathbf{n}$. Thus, $\int_C \mathbf{G} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot \mathbf{n} \, ds$. We know that $\int_C \mathbf{G} \cdot \mathbf{T} \, ds = \int_C \mathbf{G} \cdot d\mathbf{s}$. By Green's theorem this is equal to $\iint\limits_D \text{scalar curl}\, \mathbf{G} \, dA$. An easy calculation shows that $\text{scalar curl}\, \mathbf{G} = \operatorname{div} \mathbf{F}$. The planar divergence theorem follows immediately ☐

Example 13.9. Let $\mathbf{F}(x, y, z) = \begin{pmatrix} x + y + z \\ y + z \\ z \end{pmatrix}$. Let C be the ellipse parameterized by $\begin{pmatrix} 2\cos t \\ \sin t \end{pmatrix}$. Find the flux of $\mathbf{F}$ through C.

Solution: In a previous example we calculated that the area enclosed by C is 2π. The divergence of $\mathbf{F}$ is $\operatorname{div} \mathbf{F} = 3$. Thus, the flux of $\mathbf{F}$ through C is 6π.

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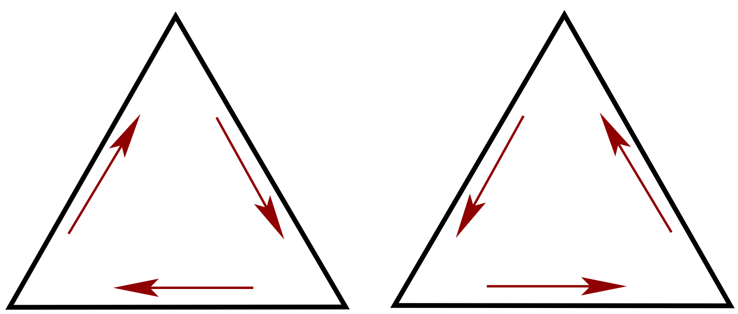
14. Surfaces: Topology and Calculus

We will study surfaces from both the topological and calculus point of view. These views don't necessarily give the same answer, but the intuition coming from topology will help us understand the calculus better.

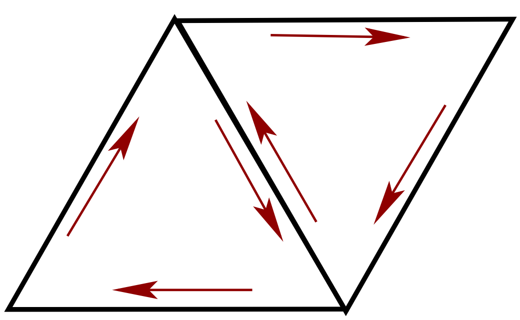
14.1. **Topological Surfaces.** Recall that a surface is a 2–dimensional manifold: that is, every point on a surface has a region around it (in the surface) that looks like the region around some point in $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$.

Examples of surfaces are spheres, tori, tori with holes in them, infinite cones, möbius bands, Klein bottles, and projective planes. It is a theorem from topology that every surface can be triangulated. (i.e. decomposed into the union of possibly bendy triangles that intersect along their sides.) We can use triangulations to help us understand the orientability of surfaces.

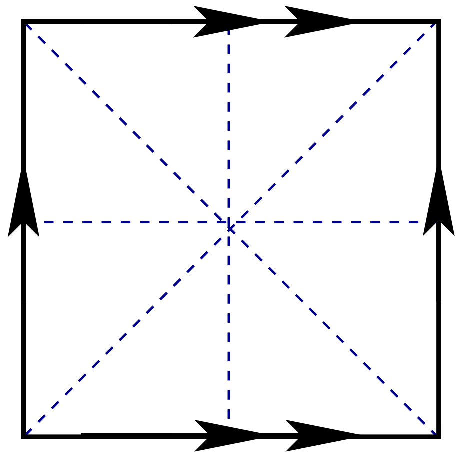
Let T be a (solid) triangle. An **orientation** on T is an orientation of ∂T. Each triangle has two possible orientations:



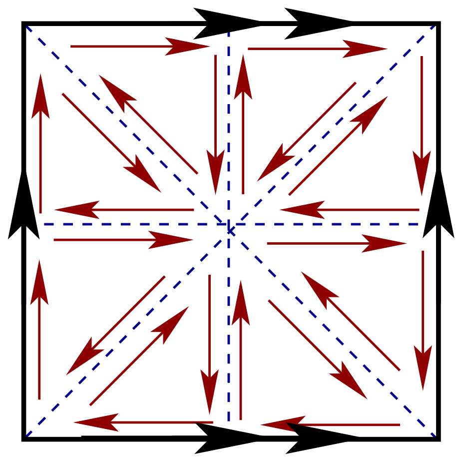
If S is a surface with triangulation $\mathcal{T}$ an **orientation** of S is a choice of orientation for each triangle in $\mathcal{T}$ such that adjacent triangles give their shared edges opposite orientations. For example:



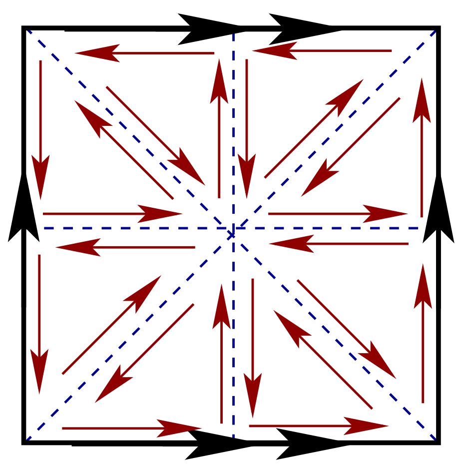
Example 14.1. Topologically, the torus can be obtained from a square by gluing opposite sides together without twisting. Here is a schematic representation of the torus with a triangulation:



Here is one possible orientation for the torus:

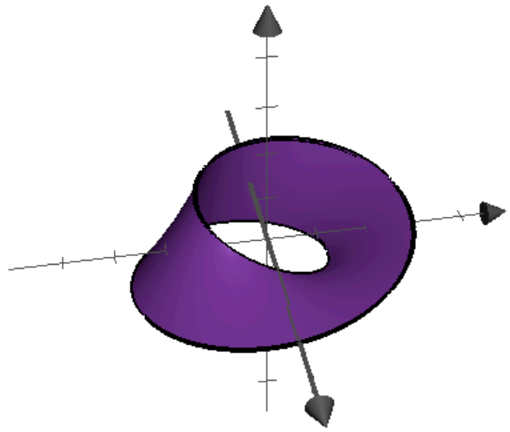


and here is another:



Not all surfaces can be oriented. If a surface S (with triangulation $\mathcal{T}$) is not orientable, we say it is **non-orientable**.

Exercise 14.2. Here is a picture of the Möbius band. Triangulate the Möbius band and prove that it is non-orientable.

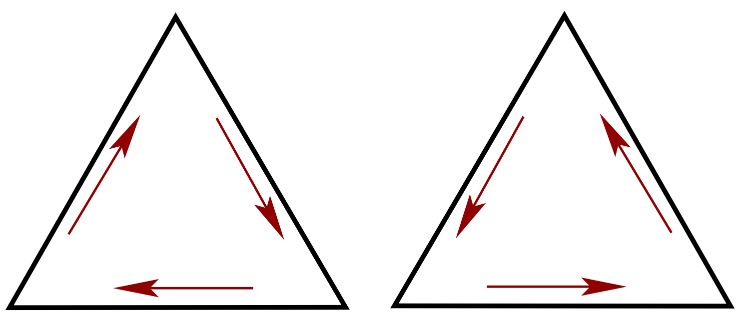


Theorem 14.3. A (topological) surface is non-orientable if and only if it contains a Möbius band as a subset.

Exercise 14.4. Prove that the Klein bottle and Projective Plane are not orientable.

If a surface S has a triangulation $\mathcal{T}$ with each triangle oriented, then we can define a unit normal vector at each point of S using the right hand rule.

For the oriented triangles below: the normal vector at each point in the triangle on the left points into the page and for each point in the triangle on the right points into the page.



If as we move from a triangle to an adjacent triangle the unit normal vector varies continuously, then we say that the choice of normal vectors is a **normal orientation**. Conversly, given a smooth surface and a unit normal vector at each point, such that the unit normal vector varies continuously across the surface we can give a triangulation of the surface an orientation. Thus, having a normal orientation is pretty much the same thing as having an orientation.

If S is a smooth surface and if we have a closed path C on S such that taking a unit normal vector to S at a point on C and pushing it around C creates a unit normal vector pointing in the opposite direction we say that S is **1-sided**. If no such path exists, then S is **2-sided**. In $\mathbb{R}^3$, a surface is 1–sided if and only if it is non-orientable. In other 3-dimensional spaces, these concepts differ. (That is, in other 3-dimensional spaces it is possible to have a 2-sided non-orientable surface.)

14.2. **Parameterized Surfaces.** Suppose that $\mathbf{X}\colon D \to \mathbb{R}^3$ is a function defined on a 2–dimensional region $D \subseteq \mathbb{R}^2$. We require that D have piece-wise C^1 boundary and that $\mathbf{X}$ be continuous and injective on the interior of D (that is on $D - \partial D$). We say that $\mathbf{X}$ is a **parameterized surface** and that it is a **parameterization** of the surface $\mathbf{X}(D) \subseteq \mathbb{R}^3$. It is possible that $\mathbf{X}(D)$ is not a surface in the topological sense.

Example 14.5. Let $\mathbf{X}(s,t) = (s,t,0)$ for $(s,t) \in D$ with D some region in $\mathbb{R}^2$.

Example 14.6. The graph of a function $z = f(x,y)$ can be parameterized as $\mathbf{X}(s,t) = (s,t,f(s,t))$.

Example 14.7. Suppose that $\mathbf{x}\colon [a,b] \to \mathbb{R}^3$ and $\mathbf{y}\colon [a,b] \to \mathbb{R}^3$ are two simple curves. Define $\mathbf{X}(s,t) = (1-s)\mathbf{x}(t) + s\mathbf{y}(t)$ for $(s,t) \in [0,1] \times [a,b]$. This is the surface of lines that joint the path $\mathbf{x}$ to the path $\mathbf{y}$.

Example 14.8. Suppose that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three non-collinear points in $\mathbb{R}^3$. The plane containing the three points can be parameterized as $\mathbf{X}(s,t) = s\mathbf{u} + t\mathbf{v} + (1-s-t)\mathbf{w}$.

Example 14.9. If $\mathbf{x}(t) = (x(t), y(t))$ is a curve in the $x-y$ plane, the surface obtained by rotating that curve around the y axis can be parameterized as

$$\mathbf{X}(s,t) = \begin{pmatrix} \cos(s)x(t) \\ y(t) \\ \sin(s)x(t) \end{pmatrix}$$

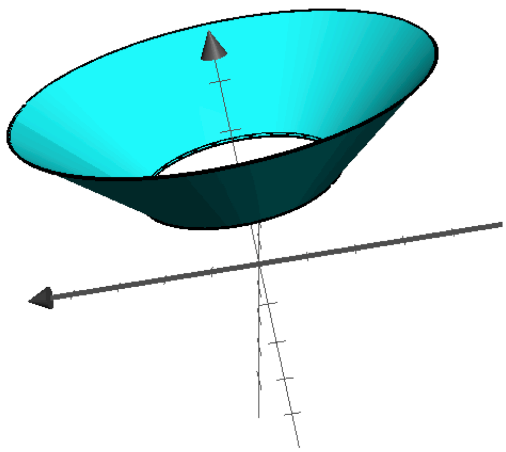
with $s \in [0, 2\pi]$.

If $\mathbf{X}\colon D \to \mathbb{R}^3$ is a surface and if we fix some t_0 then the curve $\mathbf{x}(s) = \mathbf{X}(s,t_0)$ is called a s–**coordinate curve**. Similarly, if s_0 is fixed, then $\mathbf{x}(t) = \mathbf{X}(s_0,t)$ is a t–coordinate curve. We let $\mathbf{T}_s$ and $\mathbf{T}_t$ be the tangent vectors of these curves. That is: $\mathbf{T}_s(s,t) = \frac{\partial}{\partial s}\mathbf{X}(s,t)$ and $\mathbf{T}_t(s,t) = \frac{\partial}{\partial t}\mathbf{X}(s,t)$. Notice that the vectors $\mathbf{T}_s(s,t)$ and $\mathbf{T}_t(s,t)$ are tangent to the surface $\mathbf{X}(D)$. Indeed, if $\mathbf{X}(D)$ has a tangent plane at the point (s,t) then $\mathbf{T}_s(s,t)$ and $\mathbf{T}_t(s,t)$ lie in that plane. The following definition, therefore, is likely to be useful:

If $\mathbf{X}$ is C^1 at the point (s,t), then the vector $\mathbf{N}(s,t) = \mathbf{T}_s \times \mathbf{T}_t$ is called the **normal vector** at (s,t). If $\mathbf{N}(s,t) \neq \mathbf{0}$, then we say that $\mathbf{X}$ is **smooth** at (s,t). Being smooth at (s,t) is equivalant to the statment that the vectors $\mathbf{T}_s$ and $\mathbf{T}_t$ form a basis for the tangent plane to $\mathbf{X}(D)$ at $\mathbf{X}(s,t)$.

If D is connected and if $\mathbf{X}$ is smooth, then if $\mathbf{N}$ varies continuously with (s,t), then $\mathbf{X}$ is **oriented** with orientation $\mathbf{N}/||\mathbf{N}||$. Notice that since $\mathbf{X}$ is smooth and since $\mathbf{N}$ is continuous, a connected smooth surface has exactly two orientations.

Example 14.10. Let $\mathbf{X}(s,t) = \begin{pmatrix} t\cos s \\ t \\ t\sin s \end{pmatrix}$ for $(s,t) \in [0,2\pi] \times [1,2]$. Determine if $\mathbf{X}$ is smooth and orientable.



Solution: We compute the coordinate curve derivatives as:

$$\mathbf{T}_s = \begin{pmatrix} -t\sin s \\ 0 \\ t\cos s \end{pmatrix}$$

and

$$\mathbf{T}_t = \begin{pmatrix} \cos s \\ 1 \\ \sin s \end{pmatrix}$$

A computation shows that

$$\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = \begin{pmatrix} -t\cos s \\ t \\ -t\sin s \end{pmatrix}.$$

and

$$||\mathbf{N}(s,t)|| = t\sqrt{2}$$

Since $\mathbf{N}$ is everywhere defined and is non zero for $t \in [1,2]$, $\mathbf{X}$ is smooth. Since $\mathbf{N}$ is continuous $\mathbf{X}$ is orientable.

Example 14.11. Let
$$\mathbf{X}(s,t) = \begin{pmatrix} \cos s \cos t \\ \sin t \\ sins \cos t \end{pmatrix}$$
for $(s,t) \in [0,2\pi] \times [-\pi,\pi]$. This is a parameterization of the unit sphere. Calculations show that
$$\mathbf{N}(s,t) = \begin{pmatrix} -\sin s \sin^2 t - \cos s \cos^2 t \\ -\cos^2 s \cos t \sin t - \sin^2 s \cos t \sin t \\ -\cos^2 t \sin s + \sin^2 t \cos s \end{pmatrix}.$$
It is possible to check that $\mathbf{N}$ is everywhere non-zero and continuous. Thus $\mathbf{X}$ is smooth and orientable.

14.3. **Surface Integrals.** Suppose that $\mathbf{X}\colon D \to \mathbb{R}^3$ is a smooth surface. Suppose that $f\colon \mathbf{X}(D) \to \mathbb{R}$ and $f\colon \mathbf{X}(D) \to \mathbb{R}^3$ are C^1. Then define:

$$\begin{aligned} \iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} &= \iint_D (\mathbf{F} \circ \mathbf{X}) \cdot \mathbf{N}\, dA \\ \iint_{\mathbf{X}} f\, dS &= \iint_D (f \circ \mathbf{X})\, ||\mathbf{N}||\, dA. \end{aligned}$$

Example 14.12. Let $\mathbf{X}(s,t) = \begin{pmatrix} t\cos s \\ t \\ t\sin s \end{pmatrix}$ for $(s,t) \in [0,2\pi]\times[1,2]$. Let $f(x,y,z) = x+y+z$. Compute $\iint_{\mathbf{X}} f\, dS$.

Solution: We have already calculated that

$$\mathbf{N} = \begin{pmatrix} -t\cos s \\ t \\ -t\sin s \end{pmatrix}.$$

By the formula:

$$\begin{aligned} \iint_{\mathbf{X}} f\, dS &= \iint_{[0,2\pi]\times[1,2]} f(\mathbf{X}(s,t))||\mathbf{N}(s,t)||\, dA \\ &= \int_1^2 \int_0^{2\pi} (t\cos t + t + t\sin t)t\sqrt{2}\, dsdt \\ &= \int_1^2 t^2\sqrt{2}\, dt \\ &= \tfrac{7\sqrt{2}}{3} \end{aligned}$$

Example 14.13. Let $\mathbf{X}(s,t) = \begin{pmatrix} t\cos s \\ t \\ t\sin s \end{pmatrix}$ for $(s,t) \in [0,2\pi]\times[1,2]$. Let $\mathbf{F}(x,y,z) = \begin{pmatrix} y \\ x \\ z \end{pmatrix}$. Compute $\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$.

Solution: We have already calculated that

$$\mathbf{N} = \begin{pmatrix} -t\cos s \\ t \\ -t\sin s \end{pmatrix}.$$

By the formula:

$$\begin{aligned} \iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} &= \iint_{[0,2\pi]\times[1,2]} \mathbf{F}(\mathbf{X}(s,t)) \cdot \mathbf{N}(s,t)\, dA \\ &= \int_1^2 \int_0^{2\pi} \begin{pmatrix} t \\ t\cos s \\ t\sin s \end{pmatrix} \cdot \begin{pmatrix} -t\cos s \\ t \\ -t\sin s \end{pmatrix} dsdt \\ &= \int_1^2 \int_0^{2\pi} (-t^2\cos s + t^2\cos s - t^2\sin s\, dsdt \\ &= 0 \end{aligned}$$

Example 14.14. Let $\mathbf{Y}(u,v) = \begin{pmatrix} v\cos u \\ v\sin u \\ v^2 \end{pmatrix}$ for $(u,v) \in E$ where $E = [0,2\pi] \times [0,4]$. Let $\mathbf{F}(x,y,z) = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$. Calculate $\iint_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S}$.

Recall that $\mathbf{N_Y} = \begin{pmatrix} 2v^2\cos u \\ 2v^2\sin u \\ -v \end{pmatrix}$. Thus,

$$\begin{aligned}
\iint_{\mathbf{Y}} \mathbf{F}\, d\mathbf{S} &= \iint_E \mathbf{F}(\mathbf{Y}(u,v)) \cdot \mathbf{N_Y}\, dA \\
&= \int_0^4 \int_0^{2\pi} \begin{pmatrix} -v\sin u \\ v\cos u \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2v^2\cos u \\ 2v^2\sin u \\ -v \end{pmatrix} du\, dv \\
&= \int_0^4 \int_0^{2\pi} 0\, du\, dv \\
&= 0.
\end{aligned}$$

If $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field and if $S \subset \mathbb{R}^3$ is an oriented surface, with normal orientation $\mathbf{n}$, then the **flux** of $\mathbf{F}$ across $\mathbf{S}$ is, by definition, $\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{X}$ is any parameterization of S, with normal vector $\mathbf{N}$ pointing in the same direction as $\mathbf{n}$.

Informally, the flux of $\mathbf{F}$ across S, measures the fluid flow across S.

Example 14.15. Let S be the paraboloid which is the graph of $f(x,y) = x^2 + y^2$ for $x^2 + y^2 \leq 4$. Orient S. If $\mathbf{F}(x,y,z) = (-y,x,0)$, then the flux of $\mathbf{F}$ across S is 0 since the vector field is tangent to S. (Notice that the flow lines for $\mathbf{F}$ which contain points of S, actually lie on S.

Example 14.16. Let S be the unit sphere in $\mathbb{R}^3$ with outward pointing normal. Let $\mathbf{F}(x,y,z) = (x,y,z)$. Then the flux of $\mathbf{F}$ across S is simply the surface area of S (which is 4π) since, at $(x,y,z) \in S$.

To see this, let $\mathbf{X}\colon D \to \mathbb{R}^3$ be a smooth parameterization of S with outward pointing normal vector. Noticing that $||\mathbf{F}(\mathbf{X})|| = 1$, we have:

$$\begin{aligned}
\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{X}) \cdot \mathbf{N}\, ds\, dt \\
&= \iint_D \left(\mathbf{F}(\mathbf{X}) \cdot \tfrac{\mathbf{N}}{||\mathbf{N}||}\right) ||\mathbf{N}||\, ds\, dt \\
&= \iint_D ||\mathbf{F}(\mathbf{X})||||\mathbf{N}||\, ds\, dt \\
&= \iint_D ||\mathbf{N}||\, ds\, dt \\
&= \iint_{\mathbf{X}} dS
\end{aligned}$$

and this last expression is the surface area of S.

This last example can be generalized to:

Theorem 14.17. Suppose that S is a compact surface in $\mathbb{R}^3$ and that $\mathbf{F}$ is a non-zero C^1 vector field defined in a neighborhood of S such that for each $(x, y, z) \in S$, $\mathbf{F}(x, y, z)$ is perpindicular to S. If $||\mathbf{F}(x, y, z)|| > 0$ for all $(x, y, z) \in S$, then the flux of $\mathbf{F}$ across S is simply $\pm \iint_S ||\mathbf{F}||\, dS$.

Example 14.18. Suppose that a thin sphere of radius 1 centered at the origin is given a constant $+1$ charge. Then the sphere generates an electric field given by:

$$\mathbf{E}(a, b, c) = \nabla_{(a,b,c)} \cdot \iint_S f\, dS,$$

where $f(x, y, z) = \frac{-1}{(a-x)^2+(b-y)^2+(c-z)^2}$.

We will prove that this does not depend on a parameterization for S.

14.4. **Reparameterizations.**

Definition 14.19. Suppose that D and E are 2-dimensional regions in $\mathbb{R}^2$ with C^1 boundary. Let $h\colon E \to D$ be a C^1 function such that:

(1) h is a surjection.

(2) h is one-to-one on the interior of E.

(3) Any point $\mathbf{x} \in E$ such that $\det Dh(\mathbf{x}) = 0$ lies on ∂E

Then we say that h is a **change of coordinates** function.

Example 14.20. Let D be the disc $0 \le s^2+t^2 \le 4$ in the $s-t$ plane. Let E be the rectangle $[0, 2\pi] \times [0, 2]$ in the $u - v$ plane. Define:

$$\begin{pmatrix} s \\ t \end{pmatrix} = h(u, v) = \begin{pmatrix} v\cos u \\ v \sin u \end{pmatrix}.$$

Claim: h is a change of coordinates function.

Clearly, h is a surjection and h is C^1. Notice that:

$$Dh(u, v) = \begin{pmatrix} -v\sin u & \cos u \\ v\cos u & \sin u \end{pmatrix}.$$

Thus, $\det Dh(u, v) = -v$. As long as $v > 0$, $\det Dh(u, v) \neq 0$. The set $\mathcal{P} = \{(0, v)\}$ lies in ∂E. Thus, h is a change of coordinates function.

Lemma 14.21. Suppose that E is connected and that $h\colon E \to D$ is a change of coordinates function. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are points in E at which h is C^1 and with $\det Dh(\mathbf{x}_1) \neq 0$ and $\det Dh(\mathbf{x}_2) \neq 0$, then either both $\det Dh(\mathbf{x}_1)$ and $\det Dh(\mathbf{x}_2)$ are positive, or both are negative.

Proof. There is a continuous path in E joining $\mathbf{x}_1$ to $\mathbf{x}_2$. Since h is C^1 on E, $\det Dh$ varies continuously along the path. Since the path misses the places where the determinant of the derivative of h is zero, $\det Dh(\mathbf{x}_1)$ and $\det Dh(\mathbf{x}_2)$ are both positive or both negative. □

Definition 14.22. If $h\colon E \to D$ is a change of coordinates function, and if E is connected then h is **orientation preserving** if $\det Dh > 0$ on all points where $\det Dh$ is defined and non-zero. Otherwise, h is **orientation reversing**.

Definition 14.23. Suppose that $\mathbf{X}\colon D \to \mathbb{R}^3$ is a surface and that $\mathbf{Y}\colon E \to \mathbb{R}^3$ is a surface such that there exists a change of coordinates function $h\colon E \to D$ with $\mathbf{Y} = \mathbf{X} \circ h$. Then $\mathbf{Y}$ is a reparameterization of $\mathbf{X}$.

Example 14.24. Let $\mathbf{X}(s, t) = \begin{pmatrix} s \\ t \\ s^2+t^2 \end{pmatrix}$ for $0 \le s^2 + t^2 \le 4$. Let $\mathbf{Y}(u, v) = \begin{pmatrix} v\cos u \\ v\sin u \\ v^2 \end{pmatrix}$. Notice that $\mathbf{X}$ and $\mathbf{Y}$ are parameterizations of the same paraboloid. Define $h(u, v) = \begin{pmatrix} v\cos u \\ v\sin u \end{pmatrix}$. Then $\mathbf{Y}$ is a reparameterization of $\mathbf{X}$ by an orientation reversing change of coordinates.

Lemma 14.25. Suppose that $\mathbf{X}\colon\ D \to \mathbb{R}^3$ and that $h\colon\ E \to D$ is a change of coordinates function. Let $\mathbf{Y} = \mathbf{X} \circ h$. Let $\mathbf{N_X}$ and $\mathbf{N_Y}$ be the normal vectors of $\mathbf{X}$ and $\mathbf{Y}$ respectively. Then,

$$\mathbf{N_Y}(u, v) = (\det Dh(u, v))\mathbf{N_X}(h(u, v)).$$

Proof. We simply provide a sketch for those who have taken Linear Algebra. The book provides a different method.

Let $S = \mathbf{X}(D) = \mathbf{Y}(E)$. Assume that both $\mathbf{X}$ and $\mathbf{Y}$ are smooth, so that there exists a tangent plane $TS_{\mathbf{p}}$ to S at $\mathbf{p} = \mathbf{X}(s, t) = \mathbf{Y}(u, v)$. Assume that coordinates on $\mathbb{R}^3$ have been chosen so that $TS_{\mathbf{p}}$ is the xy-plane in $\mathbb{R}^3$.

We think of $TS(u, v)$ as lying in the tangent space $T_{\mathbf{p}}$ in $\mathbb{R}^3$ at $\mathbf{p}$. Since both $\mathbf{X}$ and $\mathbf{Y}$ are smooth, the sets of vectors $\{\mathbf{T}_s, \mathbf{T}_t\}$ and $\{\mathbf{T}_u, \mathbf{T}_v\}$ are each a basis for $TS_{\mathbf{p}}$. Identifying $TS_{\mathbf{p}}$ with both the $s - t$ plane and with the $u - v$ plane.

By the chain rule,

$$D\mathbf{Y}(u, v) = D\mathbf{X}(h(u, v))Dh(u, v).$$

We have

$$\begin{aligned} D\mathbf{Y}(u, v) &= \begin{pmatrix} \mathbf{T}_u(u, v) & \mathbf{T}_v(u, v) \end{pmatrix} \\ D\mathbf{X}(h(u, v)) &= \begin{pmatrix} \mathbf{T}_s(h(u, v)) & \mathbf{T}_t(h(u, v)) \end{pmatrix} \end{aligned}$$

Recall that the absolute value of the determinant of a 2×2 matrix is the area of the parallelogram formed by its column vectors. Recall also that determinant is multiplicative. Thus, by taking determinants and absolute values we get:

(Area of parallelogram formed by $\mathbf{T}_u(u, v)$ and $\mathbf{T}_v(u, v)$) =
(Area of parallelogram formed by $\mathbf{T}_s(h(u, v))$ and $\mathbf{T}_t(h(u, v))$)$|\det Dh(u, v)|$

Thus,

$$||\mathbf{N_Y}(u, , v)|| = ||\mathbf{N_X}(h(u, v))||\,|\det Dh(u, v)|.$$

Since we have arranged that $TS_{\mathbf{p}}$ is the xy-plane, both $\mathbf{N_Y}(u, v)$ and $\mathbf{N_X}$ point in the $\pm\mathbf{k}$ direction. That is:

$$\mathbf{N_Y}(u, , v) = \begin{pmatrix} 0 \\ 0 \\ \det D\mathbf{Y}(u, v) \end{pmatrix}$$

$$\mathbf{N_X}(h(u, v)) = \begin{pmatrix} 0 \\ 0 \\ \det D\mathbf{X}(h(u, v)) \end{pmatrix}$$

Since, $\det D\mathbf{Y}(u, v) = \det D\mathbf{X}(h(u, v)) \det Dh(u, v)$, the result follows. $\square$

Thus, if $\mathbf{X}$ and $\mathbf{Y}$ are both smooth and connected surfaces and if $\mathbf{Y}$ is a reparameterization of $\mathbf{X}$ by a change of coordinates function h, then $\mathbf{Y}$ has the same normal orientation as $\mathbf{X}$ if and only if there exists a point (u, v) with $\det Dh(u, v) > 0$.

Example 14.26. Let $\mathbf{X}(s,t) = \begin{pmatrix} s \\ t \\ s^2+t^2 \end{pmatrix}$ for $0 \leq s^2+t^2 \leq 4$. Let $\mathbf{Y}(u,v) = \begin{pmatrix} v\cos u \\ v \sin u \\ v^2 \end{pmatrix}$.
Notice that $\mathbf{X}$ and $\mathbf{Y}$ are parameterizations of the same paraboloid. Define $h(u,v) = \begin{pmatrix} v\cos u \\ v\sin u \end{pmatrix}$. Notice that $\mathbf{Y} = \mathbf{X} \circ h$ where $h(u,v) = (v\cos u, v \sin u)$.

Calculations show that:

$$\begin{aligned} \mathbf{N_X} &= \begin{pmatrix} -2s \\ -2t \\ 1 \end{pmatrix} \\ \mathbf{N_Y} &= \begin{pmatrix} 2v^2 \cos u \\ 2v^2 \sin u \\ -v \end{pmatrix} \end{aligned}$$

Recalling that $\det Dh(u,v) = -v$, we see that the lemma gives us the same relationship between $\mathbf{N_X}$ and $\mathbf{N_Y}$.

You may wonder how surface integrals change under reparameterization. The following theorem provides the answer:

Theorem 14.27. Suppose that $\mathbf{X}$ and $\mathbf{Y}$ are parameterized connected surfaces and that $\mathbf{Y}$ is a reparameterization of $\mathbf{X}$. If the change of coordinate function h is orientation-preserving, let $\epsilon = +1$. If h is orientation reversing, let $\epsilon = -1$. Let f be a C^1 scalar field and let $\mathbf{F}$ be a C^1 vector field, both defined in a neighborhood of the image of $\mathbf{X}$ and $\mathbf{Y}$. Then:

$$\begin{aligned} \textstyle\iint_{\mathbf{Y}} f\, dS &= \textstyle\iint_{\mathbf{X}} f\, dS \\ \textstyle\iint_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S} &= \textstyle\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} \end{aligned}$$

Proof. We will need the change of variables theorem:

Theorem. Suppose that D and E are regions in the st plane and the uv plane respectively and that $h\colon E \to D$ is a change of coordinates function. Let $g\colon D \to \mathbb{R}$ be C^1. Then

$$\iint_E g \circ h\, |\det Dh(u,v)| du\, dv = \iint_D g\, ds\, dt$$

Both equations are a rather immediate application of this. We prove only the second, in the case when h is orientation reversing.

$$\begin{aligned} \textstyle\iint_{\mathbf{Y}} \mathbf{F}\, d\mathbf{S} &= \textstyle\iint_E (\mathbf{F} \circ \mathbf{Y}) \cdot \mathbf{N_Y}\, du\, dv \\ &= \textstyle\iint_E ((\mathbf{F} \circ \mathbf{X}) \circ h) \cdot (\mathbf{N_X} \cdot h)\, (\det Dh(u,v))\, du\, dv \\ &= \textstyle\iint_D (\mathbf{F} \circ \mathbf{X}) \cdot \mathbf{N_X}\, ds\, dt \\ &= \textstyle\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}. \end{aligned}$$

The second to last equality comes from an application of the change of variables theorem. $\square$

Definition 15.1. Suppose that S is a piecewise smooth surface which has normal orientation $\mathbf{n}$ (a unit vector). Let γ be a component of ∂S. Orient γ. We say that γ has been oriented consistently with $\mathbf{n}$ if it is possible to put a little triangle on γ, give the edges of the triangle arrows circulating in the direction of the orientation of γ, use the right hand rule and obtain a normal vector pointing in the direction of $\mathbf{n}$. We also say that ∂S has been given the orientation induced from the orientation of $\mathbf{S}$.

Example 15.2. Suppose that $A \subset \mathbb{R}^3$ is an oriented annulus (i.e. cylinder) with two boundary components. Those boundary components must have opposite orientations.

15.1. **Stokes' Theorem.**

Theorem 15.3 (Stokes' Theorem)**.** Let S be a compact, oriented, piecewise smooth surface in $\mathbb{R}^3$. Give ∂S the orientation induced by the orientation of S. Let $\mathbf{F}$ be a C^1 vector field defined on an open set containing S. Then,

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Example 15.4. Let $S \subset \mathbb{R}^3$ be the disc of radius 1 in the plane $z = 1$ with center at $(0, 0, 1)$ and normal orientation pointing in the direction of the positive z-axis. Let $\mathbf{F}(x, y, z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Compute both sides of the equality in the statement of Stokes' theorem.

Solution: We begin with the surface integral. Let D be the unit disc in the st-plane. We can parameterize S as $\mathbf{X}(s, t) = \begin{pmatrix} s \\ t \\ 1 \end{pmatrix}$ for $(s, t) \in D$. It is easy to check that the normal vector for $\mathbf{X}$ is $\mathbf{N}(s, t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Thus,

$$\begin{aligned} \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint_D 0 \, ds dt \\ &= 0 \end{aligned}$$

Now we calculate the line integral. The induced orientation on ∂S gives it the counterclockwise orientation when viewed from above. So we can parameterize ∂S as $\mathbf{x}(t) = \begin{pmatrix} \cos t \\ \sin t \\ 1 \end{pmatrix}$

with $0 \le t \le 2\pi$. Hence,

$$\begin{aligned}
\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} \begin{pmatrix} \cos t \\ \sin t \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \\
&= \int_0^{2\pi} 0\, dt \\
&= 0.
\end{aligned}$$

Example 15.5. Let $S \subset \mathbb{R}^3$ be the disc of radius 1 in the plane $z = 1$ with center at $(0, 0, 1)$ and normal orientation pointing in the direction of the positive z-axis. Let $\mathbf{F}(x, y, z) = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$. Compute both sides of the equality in the statement of Stokes' theorem.

Solution: We begin with the surface integral. Let D be the unit disc in the st-plane. We can parameterize S as $\mathbf{X}(s, t) = \begin{pmatrix} s \\ t \\ 1 \end{pmatrix}$ for $(s, t) \in D$. It is easy to check that the normal vector for $\mathbf{X}$ is $\mathbf{N}(s, t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Thus,

$$\begin{aligned}
\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dsdt \\
&= \iint_D 2\, dsdt \\
&= 2\pi
\end{aligned}$$

Now we calculate the line integral. The induced orientation on ∂S gives it the counterclockwise orientation when viewed from above. So we can parameterize ∂S as $\mathbf{x}(t) = \begin{pmatrix} \cos t \\ \sin t \\ 1 \end{pmatrix}$ with $0 \le t \le 2\pi$. Hence,

$$\begin{aligned}
\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \\
&= \int_0^{2\pi} 1\, dt \\
&= 2\pi.
\end{aligned}$$

Example 15.6. Let P be the paraboloid given by the equation $z = x^2 + y^2$ for $x^2 + y^2 \le 1$. Give P the orientation so that the normal vector to P at $\mathbf{0}$ points upward. Let $F(x, y, z)$ be the vector field $\mathbf{F}(x, y, z) = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$. Find the circulation of $\mathbf{F}$ around P.

Solution: The circulation of $\mathbf{F}$ around P is simply $\iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$. By Stokes' theorem this is equal to $\int\limits_{\partial P} \mathbf{F} \cdot d\mathbf{s}$. By the previous problem this is equal to 2π. (Notice that the orientation induced by P on $\partial P = \partial S$ is the same as the orientation induced by the disc S from the previous problem.

That example generalizes to:

Theorem 15.7. Suppose that S_1 and S_2 are two oriented surfaces such that $\partial S_1 = \partial S_2$ and such that the induced orientations on the boundary are equal. If $\mathbf{F} = \operatorname{curl} \mathbf{G}$ for some $\mathbf{G}$ then $\iint\limits_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint\limits_{S_2} \mathbf{F} \cdot d\mathbf{S}$.

Proof. By Stokes' theorem applied twice:

$$\iint\limits_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S_1} \mathbf{G} \cdot d\mathbf{s} = \int_{\partial S_2} \mathbf{G} \cdot d\mathbf{s} = \iint\limits_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

□

15.2. **Divergence Theorem.**

Theorem 15.8 (Divergence Theorem/Gauss' Theorem)**.** Let V be a compact solid region in $\mathbb{R}^3$ such that ∂V consists of piecewise smooth, closed, orientable surfaces. Orient ∂V with unit normals pointing out of V. Suppose that $\mathbf{F}$ is a C^1 vectorfield defined on an open set containing V. Then:

$$\iiint\limits_V \operatorname{div}\mathbf{F} dV = \iint\limits_{\partial V} \mathbf{F}\cdot d\mathbf{S}$$

Example 15.9. Suppose that $\mathbf{F}$ is a C^1 vector field defined on $\mathbb{R}^3 - \{\mathbf{0}\}$ such that $\operatorname{div}\mathbf{F}$ is a constant 7. Let S_1 be the unit sphere centered at the origin and let S_8 be the sphere of radius 8 centered at the origin. Orient both S_1 and S_8 outward. If the flux of $\mathbf{F}$ through S_1 is 9, find the flux of $\mathbf{F}$ through S_8.

Solution: Let V be the region between S_1 and S_8 and note that $\mathbf{F}$ is C^1 on V. Giving $\partial V = S_1 \cup S_2$ the outward orientation gives S_8 the same orientation as the one given, but gives S_1 the opposite orientation. Thus

$$\iint\limits_{\partial V} \mathbf{F}\cdot d\mathbf{S} = \iint\limits_{S_8} \mathbf{F}\cdot d\mathbf{S} - \iint\limits_{S_1} \mathbf{F}\cdot d\mathbf{S}.$$

By the divergence theorem this equals:

$$\iiint\limits_V \operatorname{div}\mathbf{F}\, dV = 7\,\mathrm{volume}(V).$$

The volume of V is simply $\frac{4\pi}{3}(8^3) - \frac{4\pi}{3} = \frac{2044\pi}{3}$. Thus,

$$\iint\limits_{S_8} \mathbf{F}\cdot d\mathbf{S} = \frac{2044\pi}{3} + \iint\limits_{S_1} \mathbf{F}\cdot d\mathbf{S} = \frac{2044\pi}{3} + 9.$$

Suppose that $\rho\colon \mathbb{R}^3 \to \mathbb{R}$ is a density function. That is, $\rho(\mathbf{x})$ is the density at a point $\mathbf{x} \in \mathbb{R}^3$. If $V \subset \mathbb{R}^3$ is a 3-dimensional region, then the mass of V is $\iiint\limits_V \rho\, dV$.

The gravitational attraction exerted by a point at $\mathbf{x}$ on another point at $\mathbf{r} \neq \mathbf{x}$ is given by:

$$\mathbf{F}(\mathbf{r}) = G\rho(\mathbf{x})\frac{\mathbf{x}-\mathbf{r}}{||\mathbf{x}-\mathbf{r}||^3}$$

where G is the universal gravitational constant.

It is easy to check that the divergence of $\mathbf{F}$ with respect to $\mathbf{r}$ is zero.

Fundamental to the study of gravitation is:

Theorem 16.1 (Gauss' Law)**.** Let V be a 3–dimensional region. The flux of the gravitational field $\mathbf{F}$ exerted by V across ∂V is:

$$\iint_{\partial V} \mathbf{F}\cdot d\mathbf{S} = -4\pi G \iiint_V \rho\, dV$$

Proof. **Simple Case:** There exists a point $\mathbf{x} \in V$ with $\rho(\mathbf{x}) \neq 0$ and all other points in V have zero density. Let S be a small sphere of radius a enclosing $\mathbf{x}$ contained inside V. Let $\mathbf{n}$ be the unit outward normal to S_a. Notice that at a point $\mathbf{r}$ on S_a, $\mathbf{n}(\mathbf{r}) = (\mathbf{r}-\mathbf{x})/a = -(\mathbf{x}-\mathbf{r})/a$.

Then

$$\begin{aligned}
\textstyle\iint_{S_a} \mathbf{F}\cdot d\mathbf{S} &= \textstyle\iint_{S_a} \mathbf{F}\cdot \mathbf{n} dS \\
&= \textstyle G\rho(\mathbf{x}) \iint_S \frac{1}{a^3}(\mathbf{x}-\mathbf{r})\cdot\frac{-1}{a}(\mathbf{x}-\mathbf{r})\, dS \\
&= \textstyle -G\rho(\mathbf{x}) \iint_S \frac{a^2}{a^4}\, dS \\
&= \textstyle -G\rho(\mathbf{x}) \iint_S \frac{1}{a^2}\, dS \\
&= \textstyle -G\rho(\mathbf{x})\frac{1}{a^2} \iint_S dS \\
&= -G\rho(\mathbf{x})(4\pi)
\end{aligned}$$

Now notice that since $\nabla_{\mathbf{r}}\cdot\mathbf{F} = 0$, by the divergence theorem, we have:

$$\iint_{\partial V} \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot d\mathbf{S} = -4\pi G\rho(\mathbf{x}).$$

The general case: For an arbitrary (integrable) density function, by superposition we have

$$\iint_{\partial V} \mathbf{F}\cdot d\mathbf{S} = -4\pi G \iiint_V \rho\, dV.$$

□

We can now prove an important theorem:

Theorem 16.2 (Shell Theorem)**.** Suppose that W is a 3–dimensional region of constant density which is the region between a sphere of radius $a \geq 0$ and a sphere of radius $b > a$, both centered at the origin.

Then the following hold:

(1) For a point $\mathbf{r}$, with $||\mathbf{r}|| > b$, the force of gravity is the same as if W were a point mass.
(2) In either case, for a point $\mathbf{r}$ with $a < ||\mathbf{r}|| < b$, the force of gravity varies linearly with distance from the origin.
(3) For a point $\mathbf{r}$ with $||\mathbf{r}|| < a$, the force of gravity is zero.

Proof. Let $\mathbf{r}$ be a point in $\mathbb{R}^3$ and let $r = ||\mathbf{r}||$. By the symmetry of W, the gravitational field at $\mathbf{r}$ is a vector that points toward the origin. That is, if $\mathbf{r} \neq \mathbf{0}$,

$$\mathbf{F}(\mathbf{r}) = -f(r)\frac{\mathbf{r}}{r}$$

where $f(r)$ is a non-negative scalar function depending only on the magnitude r of $\mathbf{r}$.

Let S be a sphere of radius r bounding a ball V centered at $\mathbf{0}$. We have:

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= f(r) \iint_S \tfrac{-\mathbf{r}}{r}\, d\mathbf{S} \\ &= -4\pi r^2 f(r). \end{aligned}$$

By Gauss' Law we also have:

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= -4\pi G \iiint\limits_V \rho\, dV \\ &= -4\pi G \operatorname{mass}(V). \end{aligned}$$

Thus,

$$-4\pi r^2 f(r) = -4\pi G \operatorname{mass}(V).$$

If $r > b$, then $\operatorname{mass}(V) = \operatorname{mass}(W)$ and so

$$f(r) = \frac{G \operatorname{mass}(W)}{r^2}$$

as desired.

If $a < r < b$, then the mass of V is equal to the mass of $W \cap V$ which varies with the cube of r. Hence $f(r)$ varies linearly with r.

If $0 < r < b$, then $\operatorname{mass}(V) = 0$ and so $f(r) = 0$. $\square$