MA 262: Practice Exam 2 **Name:**______________________

This practice exam is much longer than the actual exam.

(1) Let $\mathbf{x}(t) = (t\cos t, t\sin t)$ for $0 \leq t \leq 2\pi$ and let $F(x,y) = (-y, x)$. Find $\int_{\mathbf{x}} F \cdot d\mathbf{s}$.

(2) The gravitation vector field in $\mathbb{R}^3$ is $F(\mathbf{x}) = -\mathbf{x}/||\mathbf{x}||^3$. Find an integral representing the amount of work done by gravity as an object moves through the vector field F along the path $\mathbf{x}(t) = (t\cos t, t\sin t, t)$ for $1 \leq t \leq 2\pi$.

(3) Let $\mathbf{F}(x,y) = (x, -2y)$.

(a) Sketch a portion of the vector field F.

(b) Sketch a flow line for the vector field starting at $(1,1)$.

(c) Find a parameterization for the flow line starting at $(1,1)$.

(d) The vector field F is a gradient field. Find the potential function.

(4) Let $F(x,y) = (2xy, x^2+1)$. Find a potential function for F.

(5) Explain why flow lines for an everywhere non-zero gradient field never close up. Use this to prove that $\mathbf{F}(x,y) = (-y,x)$ is not a gradient field.

(6) Suppose that $\mathbf{F} = \nabla f$ is a C^1 gradient field on a region U. Suppose that $\gamma\colon [a,b] \to U$ is a C^1 path. Prove the Fundamental Theorem of Conservative Vector Fields which says that

$$\int_{\gamma} \mathbf{F} \cdot d\mathbf{s} = f(\gamma(b)) - f(\gamma(a)).$$

(7) Let $f(x,y) = ye^x$. Find the gradient of f.

(8) Let $\mathbf{F}(x,y,z) = (ye^x, xe^{y^2}, zx)$. Find the divergence of $\mathbf{F}$.

(9) Let $\mathbf{F}(x,y,z) = (xyz, xe^y\ln(z), x^2+y^2+z^2)$. Find the curl of $\mathbf{F}$.

(10) Find the curl of your answer to problem 7.

(11) Find the divergence of your answer to problem 9.

(12) Let $\mathbf{F}$ be a C^1 vector field. State the integral definition of the scalar curl of $\mathbf{F}$ at a point $\mathbf{a}$ and prove that it gives the same answer as the derivative definition for vector fields of the form $\mathbf{F} = (M, 0)$. You need only consider curves that are squares centered at the point $\mathbf{a}$.

(13) Explain how to prove that if a vector field $\mathbf{F}$ has path independent integrals on a region U then it is conservative.

(14) Give a complete, thorough statement of Green's theorem, including all the hypotheses on both the region and the vector field.

(15) Give a complete, thorough statement of the 2D divergence theorem, including all the hypotheses on both the region and the vector field.

(16) Prove that an irrotational vector field $\mathbf{F}$ (that is a vector field $\mathbf{F}$ with scalar curl equal to 0) on a simply connected region U is conservative.

(17) Let C be the unit circle centered at the origin in $\mathbb{R}^2$. Calculate the flux of $\mathbf{F}(x, y) = (xy, x + y^2)$ across C and also the circulation of $\mathbf{F}$ around C.

(18) Let C_1 be a square in $\mathbb{R}^2$ centered at the origin and with side length 2 and sides parallel to the axes. Let C_2 be the unit circle centered at the origin. Orient both C_1 and C_2 counterclockwise. Let $\mathbf{F}(x, y) = (2x - 5y, x + 3y + 5)$. Compute both $\int\limits_{C_1} \mathbf{F} \cdot d\mathbf{s}$ and $\int\limits_{C_2} \mathbf{F} \cdot d\mathbf{s}$.

(19) Let R be the region enclosed by the curve with parameterization $\gamma(t) = \begin{pmatrix} \cos(t + \pi)\sin(t) \\ \sin(3t) \end{pmatrix}$ for $t \in [0, \pi]$. It is shown below. Write down an integral a Calc 1 student would understand that is equal to the area of R.