**MA 262: Practice Exam 3** **Name:**________________________

These questions cover only the material since Exam 2. The first few problems are repeated from the previous practice exam.

(1) Find a parameterization of the surface formed by the graph of $z = x^2 - y^2$ with $(x,y)$ in the triangle in the $xy$-plane formed by the $x$-axis, the $y$-axis, and the line $y = -x+1$.

**Solution:** How about:

$$\mathbf{X}(s,t) = \begin{pmatrix} s \\ t \\ s^2 - t^2 \end{pmatrix}$$

with $0 \leq s \leq 1$ and $0 \leq t \leq -s+1$?

(2) Is the surface in the previous problem a smooth surface? If no, at what points is it not smooth?

**Solution:** The answer depends (somewhat) on your parameterization. The answer here is based on the parameterization above.

You can calculate that

$$\begin{aligned} \mathbf{T}_s &= (1,0,2s) \\ \mathbf{T}_t &= (0,1,-2t) \\ \mathbf{N} &= (-2s,2t,1) \end{aligned}$$

Since $\mathbf{N}$ is never $\mathbf{0}$, and since $\mathbf{X}$ is obviously $C^1$, $\mathbf{X}$ is a smooth surface.

**Solution:** How about

$$\mathbf{X}(s,t) = \begin{pmatrix} \cos s(\cos t + 5) \\ 2\sin t \\ \sin s(\cos t + 5) \end{pmatrix}$$

for $0 \leq t \leq 2\pi$ and $0 \leq s \leq 2\pi$?

(3) Consider the surface

$$\mathbf{X}(s,t) = \begin{pmatrix} 2\sin 3t + t \\ \cos 2s \\ t^2 + s^2 \end{pmatrix}, \quad 0 \leq t \leq \pi/4, \quad 0 \leq s \leq \pi$$

Find the tangent and normal vectors to $\mathbf{X}$ at the point $(\pi/6, \pi/6)$. Is the surface smooth?

**Solution:**

We have

$$\begin{aligned}\mathbf{T}_s &= (0, -2\sin 2s, 2s)\\ \mathbf{T}_t &= (6\cos(3t)+1, 0, 2t)\\ \mathbf{N} &= (-4t\sin 2s, 2s(6\cos 3t+1), 2\sin 2s(6\cos 3t+1)\end{aligned}$$

Plug $(\pi/6, \pi/6)$ into the above equations to get:

$$\begin{aligned}\mathbf{T}_s &= (0, -\sqrt{3}, \pi/3)\\ \mathbf{T}_t &= (1, 0, \pi/3)\\ \mathbf{N} &= (-\pi\sqrt{3}/3, \pi/3, \sqrt{3})\end{aligned}$$

Since $\mathbf{N}(\pi/6, \pi/6) \neq \mathbf{0}$, the surface is smooth at that point.

(4) Let $S$ be the disc of radius 1 centered at $(1,0,0)$ in $\mathbb{R}^3$ which is parallel to the $yz$-plane. Orient $S$ with normal vector pointing in the direction of the postive $x$-axis. Use the definition of surface integral to calculate the flux of $\mathbf{F}(x,y,z) = (-xy, yz, xz)$ through $S$.

**Solution:** Parameterize $S$ as:

$$\mathbf{X}(s,t) = \begin{pmatrix}1\\ s\\ t\end{pmatrix}$$

with $(s,t)$ in the region $D$ defined by $0 \leq s^2+t^2 \leq 1$. It is easy to calculate $\mathbf{N} = (1,0,0)$. Then,

$$\mathbf{F}\cdot\mathbf{N}(x,y,z) = -xy.$$

Thus, by the definition of surface integral, the flux of $\mathbf{F}$ through $S$ is

$$\iint_D \mathbf{F}\cdot\mathbf{N}(\mathbf{X}(s,t))\, dA = \iint_D -s\, ds\, dt.$$

Change to polar coordinates by setting $s = r\cos\theta$ and $t = r\sin\theta$. Then the integral above is equal to (by the change of coordinates theorem):

$$\int_0^1 \int_0^{2\pi} -r^2\cos\theta\, d\theta dr$$

Since $\int_0^{2\pi}\cos\theta d\theta = 0$, the flux equals 0.

(5) Use the same surface $S$ and $\mathbf{F}$ as in the previous problem, but now use Stoke's theorem to calculate the flux of the curl from the previous problem.

**Solution:** By Stoke's theorem,

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F}\, d\mathbf{s}.$$

Parameterize $\partial S$ as:

$$\mathbf{x}(t) = \begin{pmatrix} 1 \\ \cos t \\ \sin t \end{pmatrix}$$

with $0 \leq t \leq 2\pi$.

Notice that $\mathbf{x}$ gives $\partial S$ the orientation induced by the orientation on $S$. Then,

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{x})(t) \cdot \mathbf{x}'(t)\, dt.$$

Calculations show that this equals

$$\begin{aligned} \textstyle\int_0^{2\pi} -\cos t \sin^2 t + \sin t \cos t\, dt &= \textstyle\int_0^{2\pi} -\cos t \sin^2 t\, dt + \int_0^{2\pi} \sin t \cos t\, dt \\ &= 0. \end{aligned}$$

(6) Give precise statements of Stokes' Theorem and the Divergence Theorem.

**Stokes' Theorem:** Let $S \subset \mathbb{R}^3$ be a compact oriented piecewise $C^1$ surface such that $\partial S$ is piecewise $\mathrm{C}^1$. Give $\partial S$ the orientation induced by $S$. Suppose that $\mathbf{F}$ is a $\mathrm{C}^1$ vector field defined on $S$. Then

$$\iint\limits_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int\limits_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

**Divergence Theorem** Suppose that $V \subset \mathbb{R}^3$ is a compact 3-dimensional region with piecewise $\mathrm{C}^1$ boundary. Give $\partial V$ the orientation with outward pointing normal. If $\mathbf{F}$ is a $\mathrm{C}^1$ vector field defined throughout $V$, then

$$\iiint\limits_V \operatorname{div} \mathbf{F}\, dV = \iint_{\partial V} \mathbf{F} \cdot d\mathbf{S}.$$

(7) State and prove Gauss' law for gravity.

**Solution:** See the course notes.

(8) Use Gauss' Law for gravity and symmetry considerations to prove the shell theorem.

**Solution:** See the course notes.

(9) Suppose that a vector field $\mathbf{F}$ defined on $\mathbb{R}^3 - \{\mathbf{0}\}$ has a flux of 21 through a sphere of radius 2 (oriented outward). If the divergence of $\mathbf{F}$ is a constant $-1$, what is the flux of $\mathbf{F}$ through a sphere of radius 4 (oriented outward)?

**Solution:** Let $S_2$ be the sphere of radius 2 and let $S_4$ be the sphere of radius 4. Let $V$ be the region between them. Notice that $\mathbf{F}$ is $\mathrm{C}^1$ throughout $V$. If we give $\partial V$ the outward pointing orientation, then $S_2$ is oriented "the wrong way". Thus, by the divergence theorem:

$$-\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_4} \mathbf{F}\cdot d\mathbf{S} = \iint\limits_{\partial V} \mathbf{F}\cdot d\mathbf{S} = \iiint\limits_{V} \operatorname{div}\mathbf{F}\, dV.$$

Since $\operatorname{div}\mathbf{F} = -1$, the last integral is just the negative of the volume of $V$. The volume of $V$ is $\frac{4}{3}\pi(4)^3 - \frac{4}{3}\pi(2)^3 = 64\pi$. Thus,

$$-\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_4} \mathbf{F}\cdot d\mathbf{S} = -64$$

Since the flux through $S_2$ is 21, we have

$$\iint_{S_4} \mathbf{F}\cdot d\mathbf{S} = -64 + 21 = -43.$$

(10) Suppose that $\mathbf{F}$ is a $\mathrm{C}^1$ vector field that is everywhere tangent to the unit sphere in $\mathbb{R}^3$. Explain why the flux of $\mathbf{F}$ through the sphere must be zero. If $\mathbf{F}$ is also $\mathrm{C}^1$ everywhere inside the sphere, what can you conclude about the divergence of $\mathbf{F}$ inside the sphere?

**Solution:** Let $\mathbf{n}$ be the unit normal to the unit sphere $S$. We have:

$$\iint\limits_{S} \mathbf{F}\cdot d\mathbf{S} = \iint\limits_{S} \mathbf{F}\cdot \mathbf{n}\, dS$$

This equals 0, since $\mathbf{F}$ and $\mathbf{n}$ are always perpendicular on the sphere.

By the divergence theorem, if $\mathbf{F}$ is $\mathrm{C}^1$ inside the sphere, then the integral of the divergence of $\mathbf{F}$ over the unit ball is equal to the flux of $\mathbf{F}$ across $S$, which we just calculated to be zero. Thus, integrating the divergence of $\mathbf{F}$ over the unit ball gives zero.

(11) Suppose that $\mathbf{F}$ is a $C^1$ vector field and that $S$ is a compact surface without boundary. If the circulation of $\mathbf{F}$ around $S$ is non-zero, what can you conclude about $S$?

**Solution:** $S$ must be non-orientable. If it were orientable we could apply Stokes' theorem to conclude:

$$0 \neq \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Since $\partial S = \varnothing$, this last integral must be zero.

(12) Suppose that two surfaces $S_1$ and $S_2$ have the same oriented boundary and that they are disjoint except along their boundaries. Suppose that $\mathbf{F}$ is a $C^1$ vector field defined on the region bounded by the union of $S_1$ and $S_2$. Explain why the circulation of $\mathbf{F}$ is the same on $S_1$ and $S_2$. If the vector field is incompressible, explain why the flux through $S_1$ is the same as the flux through $S_2$.

**Solution:** By Stokes theorem, the circulation of $\mathbf{F}$ on each $S_i$ is equal to the circulation of $\mathbf{F}$ around the boundary. Since they have the same oriented boundary, they must have the same circulation.

For $\mathbf{F}$ to be incompressible, means that $\operatorname{div} \mathbf{F} = 0$. Let $S$ be the union of $S_1$ and $S_2$ without outward normal. Let $V$ be the region bounded by $S$. One of $S_1$ or $S_2$ has the wrong orientation (since they induce the same orienation on their common boundary). Thus by the divergence theorem:

$$0 = \iiint_V \operatorname{div} \mathbf{F} \, dV = \iint_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \pm \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} \mp \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

Thus, the fluxes are the same.