MA 262: Practice Exam 2 **Name:**______________________

This practice exam is much longer than the actual exam.

(1) Let $\mathbf{x}(t) = (t\cos t, t\sin t)$ for $0 \leq t \leq 2\pi$ and let $F(x, y) = (-y, x)$. Find $\int_{\mathbf{x}} F \cdot d\mathbf{s}$.

(2) The gravitation vector field in $\mathbb{R}^3$ is $F(\mathbf{x}) = -\mathbf{x}/||\mathbf{x}||^3$. Find an integral representing the amount of work done by gravity as an object moves through the vector field F along the path $\mathbf{x}(t) = (t\cos t, t\sin t, t)$ for $1 \leq t \leq 2\pi$.

Solution: We need only compute:

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_1^{2\pi} \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt.$$

We have:

$$\mathbf{F}(\mathbf{x}(t)) = \begin{pmatrix} -(t\cos t)/((t\cos t)^2 + (t\sin t)^2 + t^2)^{3/2}) \\ -(t\sin t)/((t\cos t)^2 + (t\sin t)^2 + t^2)^{3/2}) \\ -(t)/((t\cos t)^2 + (t\sin t)^2 + t^2)^{3/2}) \end{pmatrix}$$

and

$$\mathbf{x}'(t) = \begin{pmatrix} \cos t - t\sin t \\ \sin t + t\cos t \\ 1 \end{pmatrix}$$

So the integral we need to compute is:

$$\int_0^{2\pi} -\frac{1}{((t\cos t)^2 + (t\sin t)^2 + t^2)^{3/2})}\big(t\cos t(\cos t - t\sin t) + t\sin t(\sin t + t\cos t) + t\big)\, dt.$$

Notice, however, that if we were actually interested in the answer, we could use the fact that $\mathbf{F}$ is conservative and just plug in the endpoints of the path to the potential function.

(3) Let $\mathbf{F}(x, y) = (x, -2y)$.

(a) Sketch a portion of the vector field F.

(b) Sketch a flow line for the vector field starting at $(1, 1)$.

(c) Find a parameterization for the flow line starting at $(1, 1)$.

Solution: We need to find functions $\gamma(t) = (x(t), y(t))$ so that $\mathbf{F}(\gamma(t)) = \gamma'(t)$. That is, $x'(t) = x(t)$ and $y'(t) = 2y(t)$. Guessing, we see that we can use $x(t) = Ae^t$ and $y(t) = Be^{2t}$ for any $A, B \in \mathbb{R}$. We want the curve to go through $(1, 1)$ at $t = 0$, so we choose $A = 1$ and $B = 1$.

(d) The vector field F is a gradient field. Find the potential function.

Solution: Use $f(x,y) = \frac{1}{2}x^2 - y^2$.

(4) Let $F(x,y) = (2xy, x^2+1)$. Find a potential function for F.

Solution: Integrate $2xy$ with respect to x to obtain $f(x,y) = x^2y + h(y)$ where $h(y)$ is any real valued differentiable function of y. Take the derivative with respect to y to get

$$x^2 + \frac{\partial}{\partial y}h(y) = x^2 + 1.$$

Thus, $h(y) = y + c$ for any constant c. We might as well choose $c = 0$. So we find that $f(x,y) = x^2y + y$ is a potential function for $\mathbf{F}$.

(5) Explain why flow lines for an everywhere non-zero gradient field never close up. Use this to prove that $\mathbf{F}(x,y) = (-y, x)$ is not a gradient field.

Solution: If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a potential function, the fundamental theorem of conservative vector fields says

$$\int\limits_\gamma \nabla f \cdot d\mathbf{s} = f(\text{end}) - f(\text{start}),$$

where start and end are the start and end points of γ. Thus, if γ is a closed loop, $\int\limits_\gamma \nabla f \cdot d\mathbf{s} = 0$.

(6) Suppose that $\mathbf{F} = \nabla f$ is a C^1 gradient field on a region U. Suppose that $\gamma\colon [a,b] \to U$ is a C^1 path. Prove the Fundamental Theorem of Conservative Vector Fields which says that

$$\int\limits_\gamma \mathbf{F} \cdot d\mathbf{s} = f(\gamma(b)) - f(\gamma(a)).$$

Solution: Suppose that $\mathbf{F} = \nabla f$ is a C^1 conservative vector field. By the chain rule,

$$\frac{d}{dt}f(\gamma(t)) = Df(\gamma(t))\gamma'(t).$$

Recall that since f is a scalar field, Df is the transpose of ∇f, so

$$\frac{d}{dt}f(\gamma(t)) = \mathbf{F}(\gamma(t)) \cdot \gamma'(t).$$

Thus,

$$\int\limits_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\, dt = \int_a^b \frac{d}{dt}f(\gamma(t))\, dt.$$

By the fundamental theorem of calculus, this equals $f(\gamma(b)) - f(\gamma(a))$ as desired.

(7) Let $f(x,y) = ye^x$. Find the gradient of f.

Solution: $\nabla f(x,y) = \begin{pmatrix} ye^x \\ e^x \end{pmatrix}$.

(8) Let $\mathbf{F}(x,y,z) = (ye^x, xe^{y^2}, zx)$. Find the divergence of $\mathbf{F}$.

Solution:

$$\operatorname{div}\mathbf{F}(x,y,z) = ye^x + 2yxe^{y^2} + x$$

(9) Let $\mathbf{F}(x,y,z) = (xyz, xe^y\ln(z), x^2+y^2+z^2)$. Find the curl of $\mathbf{F}$.

Solution:

$$\operatorname{curl}\mathbf{F}(x,y,z) = \begin{pmatrix} 2y - xe^y/z \\ -(2x - xz) \\ e^y\ln z - xz \end{pmatrix}$$

(10) Find the curl of your answer to problem 7.

Solution: It is (0,0,0) because the curl of every gradient field is zero.

(11) Find the divergence of your answer to problem 9.

Solution: It is 0 because the divergence of every rotational vector field is 0.

(12) Let $\mathbf{F}$ be a C^1 vector field. State the integral definition of the scalar curl of $\mathbf{F}$ at a point $\mathbf{a}$ and prove that it gives the same answer as the derivative definition for vector fields of the form $\mathbf{F} = (M, 0)$. You need only consider curves that are squares centered at the point $\mathbf{a}$.

Solution: Let C_n be the square of side length $2/n$ centered at $\mathbf{a} = (a_1, a_2)$. The scalar curl of $\mathbf{F}$ is defined to be:

$$\lim \frac{1}{\text{area}(C_n)} \int\limits_{C_n} \mathbf{F}\cdot d\mathbf{s}.$$

Notice the curves C_n enclose and converge to $\mathbf{a}$. We orient each of them counterclockwise. The term area(C_n) denotes the area enclosed by C_n. In this case, it is $4/n^2$.

Assume that $\mathbf{F} = (M, 0)$. Notice that it is perpendicular to the vertical sides of C_n, so its integral over those sides will be 0. We parametrize the other two sides as $\gamma(t) = (a_1 + t, a_2 - 1/n)$ and $\psi(t) = (a_1 + t, a_2 + 1/n)$, both for $t \in [-1/n, 1/n]$. Observe that $\gamma'(t) = (1,0)$ and $\psi'(t) = (1,0)$, so γ is parameterized in the correct direction and ψ in the wrong direction. Thus,

$$\int\limits_{C_n} \mathbf{F}\cdot d\mathbf{s} = \int\limits_{-1/n}^{1/n} \mathbf{F}(\gamma(t)) \cdot \begin{pmatrix}1\\0\end{pmatrix} - \mathbf{F}(\psi(t)) \cdot \begin{pmatrix}1\\0\end{pmatrix}\, dt.$$

This equals

$$\int_{-1/n}^{1/n} M(a_1 + t, a_2 - 1/n) - M(a_1 + t, a_2 + 1/n)\, dt.$$

(13) Explain how to prove that if a vector field $\mathbf{F}$ has path independent integrals on a region U then it is conservative.

Solution: We assume that U is path connected. If it isn't apply the following argument to each path-connected piece. Choose a basepoint $\mathbf{a} \in U$. For each $x \in U$, let γ_x be a piecewise C^1 path from $\mathbf{a}$ to x. Define $f(x) = \int_{\gamma_x} \mathbf{F} \cdot d\mathbf{s}$. Since $\mathbf{F}$ has path-independent integrals the value of $f(x)$ doesn't depend on what path we choose, only on our choice $\mathbf{a}$ and, of course, on x.

We then need to show that f is differentiable and that $\nabla f = \mathbf{F}$. To do this, suppose that $\mathbf{F} = (M, N)$. We show that $\frac{\partial}{\partial x} f = M$ and $\frac{\partial}{\partial x} f = N$. Since M and N are continuous on U, a theorem from MA 122 guarantees that f is differentiable and $\nabla f = \mathbf{F}$.

Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. By definition

$$\frac{\partial}{\partial x} f(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + he_1) - f(\mathbf{x})}{h}.$$

Choose a path $\gamma_{\mathbf{x}}$ from $\mathbf{a}$ to $\mathbf{x}$. Let $L(h)$ be a horizontal oriented line segment from $\mathbf{x}$ to $\mathbf{x} + he_1$. Let ψ be the path $\gamma_{\mathbf{x}}$ followed by the path $L(h)$. Then

$$f(\mathbf{x}) = \int_{\gamma_{\mathbf{x}}} \mathbf{F} \cdot d\mathbf{s}$$

and

$$f(\mathbf{x} + he_1) = \int_{\psi} \mathbf{F} \cdot d\mathbf{s} = \int_{\gamma_{\mathbf{x}}} \mathbf{F} \cdot d\mathbf{s} + \int_{L(h)} \mathbf{F} \cdot d\mathbf{s}$$

Thus,

$$f(\mathbf{x} + he_1) - f(x) = \int_{L(h)} \mathbf{F} \cdot d\mathbf{s}.$$

Notice that parameterize $L(h)$ by $\ell(t) = \mathbf{x} + the_1$ for $t \in [0, 1]$. Then $\ell'(t) = he_1$. Thus,

$$\frac{\partial}{\partial x} f(\mathbf{x}) = \lim_{h \to 0} \frac{1}{h} \int_0^1 M(\ell(t))h\, dt = \lim_{h \to 0} \int_0^1 M(\ell(t))\, dt$$

By the Mean Value Theorem for Integrals, there exists $t^* \in [0, 1]$ so that

$$\int_0^1 M(\ell(t))\, dt = M(\ell(t^*)).$$

As $h \to 0$, by continuity of M, $M(\ell(t^*)) \to M(\ell(0)) = M(\mathbf{x})$.

The computation for $\frac{\partial}{\partial y} f(\mathbf{x}) = N(\mathbf{x})$ is similar.

(14) Give a complete, thorough statement of Green's theorem, including all the hypotheses on both the region and the vector field.

Solution: Suppose that $U \subset \mathbb{R}^2$ is open and that $\mathbf{F}$ is a C^1 vector field defined on U. Suppose that $D \subset U$ is a 2-dimensional region such that ∂D is a finite collection of piecewise C^1 simple closed curves, each oriented so that D is on the left. Then

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D \operatorname{scurl} \mathbf{F} \, dA$$

(15) Give a complete, thorough statement of the 2D divergence theorem, including all the hypotheses on both the region and the vector field.

Solution: Suppose that $U \subset \mathbb{R}^2$ is open and that $\mathbf{F}$ is a C^1 vector field defined on U. Suppose that $D \subset U$ is a 2-dimensional region such that ∂D is a finite collection of piecewise C^1 simple closed curves, each oriented so that D is on the left. Then

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{n} \cdot ds = \iint_D \operatorname{div} \mathbf{F} \, dA$$

where $\mathbf{n}$ is the obtained by rotating the unit tangent vector to the right by $\pi/2$ radians (i.e. the outward pointing unit normal.)

(16) Prove that an irrotational vector field $\mathbf{F}$ (that is a vector field $\mathbf{F}$ with scalar curl equal to 0) on a simply connected region U is conservative.

Solution: We use Green's theorem. Let $C \subset U$ be a simple closed curve. Since U is simply connected, by the Jordan Curve Theorem, C bounds a region $D \subset U$. By Green's theorem (taking into account the orientation of C):

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \pm \int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \pm \iint_D \operatorname{scurl} \mathbf{F} \, dA = \pm \iint_D 0 \, dA = 0.$$

Since the integral of $\mathbf{F}$ around any simple closed curve is zero, we have theorem that tells us that $\mathbf{F}$ must be conservative.

(17) Let C be the unit circle centered at the origin in $\mathbb{R}^2$. Calculate the flux of $\mathbf{F}(x, y) = (xy, x + y^2)$ across C and also the circulation of $\mathbf{F}$ around C.

Solution: Parameterize C by $C(t) = (\cos t, \sin t)$. Observe that $C'(t) = (-\sin t, \cos t)$ and that the outward pointing unit normal is $\mathbf{n}(t) = (\cos(t), \sin(t))$. The flux of $\mathbf{F}$ across C is

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds.$$

By the 2D divergence theorem, with D being the unit disc and $\operatorname{div} \mathbf{F}(x, y) = 3y$ we have the flux equal to

$$\iint_D 3y \, dA.$$

Since the function $f(x, y) = y$ has the property that $f(-x, -y) = -f(x, y)$ and since D is symmetric across the origin, this integral equal zero.

The circulation of $\mathbf{F}$ around C is given by the integral

$$\int_C \mathbf{F} \cdot d\mathbf{s}.$$

By Green's theorem and the fact that $\operatorname{scurl}\mathbf{F}(x,y) = 1 - x$ we have that circulation is equal to

$$\iint_D \operatorname{scurl} 1 - x\, dA = \iint_D 1\, dA - \iint_D x\, dA.$$

The same symmetry argument as before shows that the second integral is zero, while the first integral is just the area of the disc. So the circulation of $\mathbf{F}$ around C is 2π.

(18) Let C_1 be a square in $\mathbb{R}^2$ centered at the origin and with side length 2 and sides parallel to the axes. Let C_2 be the unit circle centered at the origin. Orient both C_1 and C_2 counterclockwise. Let $\mathbf{F}(x,y) = (2x - 5y, x + 3y + 5)$. Compute both $\int_{C_1} \mathbf{F} \cdot d\mathbf{s}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{s}$.

Solution: Observe that $\operatorname{scurl}\mathbf{F}(x,y) = 1 + 5 = 6$. Applying Green's theorem to the regions D_1 and D_2 bounded by C_1 and C_2 respectively shows that the integrals are 6π and 24 respectively.

(19) Let R be the region enclosed by the curve with parameterization $\gamma(t) = \begin{pmatrix} \cos(t+\pi)\sin(t) \\ \sin(3t) \end{pmatrix}$. It is shown below. Write down an integral a Calc 1 student would understand that is equal to the area of R.

Set $\mathbf{F}(x,y) = \begin{pmatrix} -y \\ 0 \end{pmatrix}$ and observe that $\operatorname{scurl}\mathbf{F}(x,y) = 1$. By Green's theorem, the area is

$$\iint_R 1\, dA = \iint_R \operatorname{scurl}\mathbf{F}\, dA = \pm \int_\gamma \mathbf{F} \cdot d\mathbf{s},$$

where the $\pm$ is determined by taking into account whether or not γ is oriented so that $R$is on its left.

Now $\mathbf{F}(\gamma(t)) = (-\sin(3t), 0)$ and $\gamma'(t) = \begin{pmatrix} -\sin(t+\pi)\sin(t) + \cos(t+\pi)\cos(t) \\ 3\cos(3t) \end{pmatrix}$. Thus, the area is equal to

$$\left| \int_0^\pi -\sin(3t)\Big(-\sin(t+\pi)\sin(t) + \cos(t+\pi)\cos(t)\Big)\, dt \right|.$$

There are other possible solutions here, depending what vector field $\mathbf{F}$ was chosen. If you are interested, the answer is 6/5.