MA 262: Practice Exam 1 Solutions **Name:**________________________

This practice exam is much longer than the actual exam.

(1) Let $F(x,y) = (x^2y, y^2x, 3x - 2yx)$. Find the derivative of F.

Solution:

$$DF(x,y) = \begin{pmatrix} 2xy & x^2 \\ y^2 & 2yx \\ 3-2y & -2x \end{pmatrix}$$

(2) Let $F(x,y) = (x-y, x+y)$ and let $G(x,y) = (x\cos y, x\sin y)$. Find the derivative of $F \circ G$ using the chain rule.

Solution:

$$\begin{aligned} DF(x,y) &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ DG(x,y) &= \begin{pmatrix} \cos y & -x\sin y \\ \sin y & x\cos y \end{pmatrix} \\ D(F\circ G)(x,y) &= DF(G(x,y))DG(x,y) \\ &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \cos y & -x\sin y \\ \sin y & x\cos y \end{pmatrix} \\ &= \begin{pmatrix} \cos y - \sin y & -x\sin y - x\cos y \\ \cos y + \sin y & -x\sin y + x\cos y \end{pmatrix} \end{aligned}$$

(3) Suppose that a rotating circle of radius 1 is travelling through the plane, so that at time t seconds the center of the circle is at the point $(t, \sin t)$. Let P be the point on the circle which is at $(0,1)$ at time $t = 0$. If the circle makes 3 revolutions per second, what is the path $\mathbf{x}(t)$ taken by the point P?

Solution: The rotation of the P relative to the center of the circle (that is, in $T_{\mathbf{c}(t)}$) can be described by the path $(\cos(6\pi t + \pi/2), \sin(6\pi t + \pi/2))$. Thus, $\mathbf{x}(t) = (\cos(6\pi t + \pi/2) + t, \sin(6\pi t + \pi/2) + \sin t)$.

(4) A rotating circle of radius 1 follows a helical path in $\mathbb{R}^3$ so that at time t the center of the circle is at $(\sin t, \cos t, t)$. At each time t, the circle lies in the osculating plane. (That is, the circle lies in the plane spanned by the unit tangent and the unit normal vectors.) Let P be the point on the circle which is at $(1,0)$ at time $t = 0$. The circle completes one rotation every 2π seconds. Find a formula $\mathbf{x}(t)$ for the path taken by the point P. (Hint: Express the center of the circle as a combination of the unit tangent and normal vectors.)

Solution: Relative to the center of the circle (that is, in $T_{\mathbf{c}(t)}$) the point P follows the path $\cos t\mathbf{T} + \sin t\mathbf{N}$ where $\mathbf{T}$ and $\mathbf{N}$ are the unit tangent and unit normal vectors to $\mathbf{c}(t) = (\sin t, \cos t, t)$ respectively. Those formulae are

$$\begin{aligned} \mathbf{T}(t) &= \tfrac{1}{\sqrt{2}} \begin{pmatrix} \cos t \\ -\sin t \\ 1 \end{pmatrix} \\ \mathbf{N}(t) &= \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix} \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{c}(t) &= \cos t\mathbf{T} + \sin t\mathbf{N} + \mathbf{c}(t) \\ &= \tfrac{\cos t}{\sqrt{2}} \begin{pmatrix} \cos t \\ -\sin t \\ 1 \end{pmatrix} + \sin t \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix} + \begin{pmatrix} \sin t \\ \cos t \\ t \end{pmatrix}. \end{aligned}$$

(5) Use the Mean Value Theorem to explain the relationship between calculating arc length of a C^1 curve $\gamma\colon [a, b] \to \mathbb{R}^2$ by taking the limit of polygonal approximations and the integral $\int_a^b ||\gamma'(t)||\, dt$.

Solution: Subdivide the interval $[a, b]$ using

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

Assume, for convenience that they are equally spaced so that $t_{i+1} - t_i = \Delta t$ for every i. The points $\gamma(t_i)$ lie on the curve and $||\gamma(t_{i+1}) - \gamma(t_i)||$ is the length of the line segment between $\gamma(t_{i+1})$ and $\gamma(t_i)$. As $\Delta t \to 0$ (equivalently as $n \to \infty$) the union of those line segments approaches the curve. The length of the curve is then

$$\lim_{\Delta t \to 0} \sum_{i=0}^{i=1} ||\gamma(t_{i+1}) - \gamma(t_i)||.$$

Write $\gamma(t) = (x(t), y(t))$. Since x and y are C^1 functions, by the mean value theorem there exist $t_i^*, t_i^{**} \in [t_i, t_{i+1}]$ so that

$$\begin{aligned} x'(t_i^*)\Delta t &= x(t_{i+1}) - x(t_i) \\ y'(t_i^{**})\Delta t &= y(t_{i+1}) - y(t_i) \end{aligned}$$

Thus,

$$||\gamma(t_{i+1}) - \gamma(t_i)|| = ||(x'(t_i^*), y'(t_i^{**})||\Delta t.$$

As $\Delta \to 0$, we have $t_i^*, t_i^{**} \to t_i$. Assuming that they converge faster than the sum of the errors accumulates, we have:

$$\lim_{\Delta t \to 0} \sum_{i=0}^{i=1} ||\gamma(t_{i+1}) - \gamma(t_i)|| = \lim_{\Delta t \to 0} \sum_{i=0}^{i=1} ||\gamma'(t_i)||\Delta t.$$

We recognize this latter quantity as a Riemann sum and so the length of γ is equal to

$$\int_a^b ||\gamma'(t)||\, dt.$$

(6) Explain what it means for curvature to be an intrinsic quantity.

Solution: The curvature of a path $\mathbf{x}(t)$ at t_0, depends only on the curve itself at t_0, not on the parameterization $\mathbf{x}$.

(7) Prove that the curvature at any point of a circle of radius r is $1/r$.

Solution: A circle of radius r can be parameterized as $\mathbf{x}(t) = (r\cos t, r\sin t)$ for $0 \leq t \leq 2\pi$. We have:

$$\begin{aligned}
\mathbf{x}'(t) &= (-r\sin t, r\cos t) \\
||\mathbf{x}'(t)|| &= r \\
\mathbf{T} &= (-\sin t, \cos t) \\
\mathbf{T}' &= (-\cos t, -\sin t) \\
||\mathbf{T}'|| &= 1 \\
\kappa(t) &= ||\mathbf{T}'||/||\mathbf{x}'|| \\
&= 1/r.
\end{aligned}$$

(8) Prove that the curvature at any point of a line is 0.

Solution: We can parameterize a line segment as

$$\gamma(t) = \mathbf{a} + t\mathbf{b}$$

for some $\mathbf{a}$, and non-zero $\mathbf{b}$ in $\mathbb{R}^n$. We see that

$$\gamma'(t) = \mathbf{b}.$$

Thus, $\mathbf{T} = \mathbf{b}/||\mathbf{b}||$ and $\mathbf{T}' = 0$. Thus, $\kappa = ||\mathbf{T}'||/||\mathbf{b}|| = 0$.

(9) Let $\mathbf{x}(t) = (\cos t, \sin t, t)$ for $1 \leq t \leq 2$. Find $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ (that is, the moving frame) for $\mathbf{x}$ and also find κ (the curvature).

Solution: We have:

$$\begin{aligned}
\mathbf{x}'(t) &= (-\sin t, \cos t, t) \\
||\mathbf{x}'(t)|| &= \sqrt{2} \\
\mathbf{T}(t) &= \tfrac{1}{\sqrt{2}}(-\sin t, \cos t, 1) \\
\mathbf{T}'(t) &= \tfrac{1}{\sqrt{2}}(-\cos t, -\sin t, 0) \\
\mathbf{N}(t) &= (-\cos t, -\sin t, 0) \\
\mathbf{B}(t) &= \tfrac{1}{\sqrt{2}}(\sin t, -\cos t, 1) \\
\kappa(t) &= 1/2
\end{aligned}$$

(10) Suppose that $\mathbf{x}\colon [a,b] \to \mathbb{R}^n$ is a C^1 path such that for all t, $||\mathbf{x}(t)|| = 5$. Prove that at each t, $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ are perpendicular.

Solution: Since $5 = \mathbf{x}(t) \cdot \mathbf{x}(t)$, by the product rule, we have $0 = 2\mathbf{x} \cdot \mathbf{x}'$, implying that $\mathbf{x}$ and $\mathbf{x}'$ are perpendicular.

(11) A particle is following the path $\mathbf{x}(t) = (t, t^2, t^3)$ for $1 \leq t \leq 5$. Find an integral representing the distance travelled by the particle after t seconds.

Solution: The distance travelled after t seconds is

$$\begin{aligned} s(t) &= \int_1^t ||\mathbf{x}'(\tau)||\, d\tau \\ &= \int_1^t \sqrt{1+4t^2+9t^4}\, d\tau. \end{aligned}$$

(12) Let $\mathbf{x}(t) = (t^2, 3t^2)$ for $t \geq 1$. Reparameterize $\mathbf{x}$ by arc length.

Solution: We compute,

$$s(t) = \int_1^t \sqrt{4\tau^2 + 36\tau^2}\, d\tau = \int_1^t 2\tau\sqrt{10}\, d\tau. = \sqrt{10}(t^2 - 1).$$

Then,

$$s^{-1}(t) = \sqrt{t/\sqrt{10} + 1}$$

Consequently,

$$\mathbf{y}(t) = \mathbf{x} \circ s^{-1}(t) = (t/\sqrt{10} + 1, 3t/\sqrt{10} + 3)$$

is the reparameterization of $\mathbf{x}$ by arclength.

(13) Suppose that $\mathbf{x}(t)$ is a path in $\mathbb{R}^n$ such that $\mathbf{x}(0) = \mathbf{a}$ and $\mathbf{x}(1) = \mathbf{b}$ (that is, $\mathbf{x}$ is a path joining $\mathbf{a}$ to $\mathbf{b}$.) Find a path which has the same image as $\mathbf{x}$ but which joins $\mathbf{b}$ to $\mathbf{a}$.

Solution: $\mathbf{y}\colon [-1, 0] \to \mathbb{R}^n$ defined by $\mathbf{y}(t) = \mathbf{x}(-t)$ will do the trick since $\mathbf{y}(0) = \mathbf{a}$ and $\mathbf{y}(-1) = \mathbf{b}$.

(14) Let $\mathbf{x}\colon [a, b] \to \mathbb{R}^n$ be a path with $\mathbf{x}'(t) \neq \mathbf{0}$ for all t. Let $\mathbf{y} = \mathbf{x} \circ \phi$ be an orientation reversing reparameterization of $\mathbf{x}$. Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is integrable. Prove that $\int_{\mathbf{y}} f\, ds = \int_{\mathbf{x}} f\, ds$.

Solution: Since ϕ is orientation reversing, $|\phi'(t)| = -\phi'(t)$. Hence, $||\mathbf{y}'(t)|| = -||\mathbf{x}'(\phi(t))||\phi'(t)$. Thus,

$$\int_{\mathbf{y}} f\, ds = -\int_c^d f(\mathbf{x}(\phi(t)))||\mathbf{x}'(\phi(t))||\phi'(t)\, dt.$$

Substitute $u = \phi(t)$ and $du = \phi'(t)dt$ to get:

$$\int_{\mathbf{y}} f\, ds = -\int_b^a f(\mathbf{x}(u))||\mathbf{x}'(u)||\, du.$$

Reversing the limits of integration eliminates the negative sign and so the result follows.

(15) Let $\mathbf{x}(t) = (t\cos t, t\sin t)$ for $0 \leq t \leq 2\pi$. Let $f(x, y) = y\cos x$. Let $F(x, y) = (-y, x)$. Find one-variable integrals representing $\int_{\mathbf{x}} f\, ds$ and $\int_{\mathbf{x}} F \cdot d\mathbf{s}$.

Solution: Notice that

$$\begin{aligned} \mathbf{x}(t) &= (t\cos t, t\sin t) \\ \mathbf{x}'(t) &= (\cos t - t\sin t, t\cos t + \sin t) \\ ||\mathbf{x}'(t)|| &= \sqrt{(\cos t - t\sin t)^2 + (t\cos t + \sin t)^2} \end{aligned}$$

Thus,

$$\int_{\mathbf{x}} f\, ds = \int_0^{2\pi} t\sin t\cos(t\cos t)\sqrt{(\cos t - t\sin t)^2 + (t\cos t + \sin t)^2}\, dt.$$

And,

$$
\begin{aligned}
\int_{\mathbf{x}} F \cdot ds &= \int_0^{2\pi} \begin{pmatrix} -t \sin t \\ t \cos t \end{pmatrix} \cdot \begin{pmatrix} \cos t - t \sin t \\ \sin t + t \cos t \end{pmatrix} \, dt. \\
&= \int_0^{2\pi} t^2 \, dt \\
&= 8\pi^3/3.
\end{aligned}
$$