MA 262: Practice Exam 1 **Name:**________________________

This practice exam is much longer than the actual exam.

(1) Let $F(x, y) = (x^2y, y^2x, 3x - 2yx)$. Find the derivative of F.

(2) Let $F(x, y) = (x - y, x + y)$ and let $G(x, y) = (x \cos y, x \sin y)$. Find the derivative of $F \circ G$ using the chain rule.

(3) Suppose that a rotating circle of radius 1 is travelling through the plane, so that at time t seconds the center of the circle is at the point $(t, \sin t)$. Let P be the point on the circle which is at $(0, 1)$ at time $t = 0$. If the circle makes 3 revolutions per second, what is the path $\mathbf{x}(t)$ taken by the point P?

(4) A rotating circle of radius 1 follows a helical path in $\mathbb{R}^3$ so that at time t the center of the circle is at $(\sin t, \cos t, t)$. At each time t, the circle lies in the osculating plane. (That is, the circle lies in the plane spanned by the unit tangent and the unit normal vectors.) Let P be the point on the circle which is at $(1, 0)$ at time $t = 0$. The circle completes one rotation every 2π seconds. Find a formula $\mathbf{x}(t)$ for the path taken by the point P. (Hint: Express the center of the circle as a combination of the unit tangent and normal vectors.)

(5) Use the Mean Value Theorem to explain the relationship between calculating arc length of a C^1 curve $\gamma\colon [a, b] \to \mathbb{R}^2$ by taking the limit of polygonal approximations and the integral $\int_a^b ||\gamma'(t)||\, dt$.

(6) Explain what it means for curvature to be an intrinsic quantity.

(7) Prove that the curvature at any point of a circle of radius r is $1/r$.

(8) Prove that a straight line has zero curvature.

(9) Let $\mathbf{x}(t) = (\cos t, \sin t, t)$ for $1 \leq t \leq 2$. Find $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ (that is, the moving frame) for $\mathbf{x}$ and also find κ (the curvature).

(10) Suppose that $\mathbf{x}\colon [a, b] \to \mathbb{R}^n$ is a C^1 path such that for all t, $||\mathbf{x}(t)|| = 5$. Prove that at each t, $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ are perpendicular.

(11) A particle is following the path $\mathbf{x}(t) = (t, t^2, t^3)$ for $1 \leq t \leq 5$. Find an integral representing the distance travelled by the particle after t seconds.

(12) Let $\mathbf{x}(t) = (t^2, 3t^2)$ for $t \geq 1$. Reparameterize $\mathbf{x}$ by arc length.

(13) Suppose that $\mathbf{x}(t)$ is a path in $\mathbb{R}^n$ such that $\mathbf{x}(0) = \mathbf{a}$ and $\mathbf{x}(1) = \mathbf{b}$ (that is, $\mathbf{x}$ is a path joining $\mathbf{a}$ to $\mathbf{b}$.) Find a path which has the same image as $\mathbf{x}$ but which joins $\mathbf{b}$ to $\mathbf{a}$.

(14) Let $\mathbf{x}\colon [a,b] \to \mathbb{R}^n$ be a path with $\mathbf{x}'(t) \neq \mathbf{0}$ for all t. Let $\mathbf{y} = \mathbf{x} \circ \phi$ be an orientation reversing reparameterization of $\mathbf{x}$. Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is integrable. Prove that $\int_{\mathbf{y}} f\, ds = \int_{\mathbf{x}} f\, ds$.

(15) Let $\mathbf{x}(t) = (t\cos t, t\sin t)$ for $0 \leq t \leq 2\pi$. Let $f(x,y) = y\cos x$. Find a one-variable integral representing $\int_{\mathbf{x}} f\, ds$.