MA 262 Homework 7: Harmonic Functions sound good to me!

The purpose of this assignment is two-fold. Firsty, you will learn about harmonic functions, their relationships to the vector calculus concepts we've been studying and some applications. Second, you will begin to study surface parameterizations with a view to learning higher dimensional versions of Green's theorem. Remember that you are encouraged to work with others, but you should give credit to everyone who helps you!

You are encouraged to make significant progress on 7A - 7C prior to the exam.

Problem 7.A. Read the article "Circles in Circles: Creating a mathematical model of surface water waves" by Katherine Socha. (*Math. Month.* 114, March 2007). Answer the following briefly, but thoroughly. Yes, really do just take the time to read the article before jumping into the questions.

(1) Summarize the main goal of the article.

(2) Summarize how differential equations are related to the goal of the article.

(3) Socha makes three assumptions about the behaviour of water. For each explain the connection (if any) to concepts from our class.

(4) If you were in conversation with Socha about her paper, how might you respond to the statement on page 206 that begins "By a standard theorem from vector calculus ... " How might she respond to your comments?

(5) What role in the model is played by our understanding of waves as "wavy"? (i.e. why is it important that some variation of cosine be used to represent the wave?)

(6) What is the significance of the function Z and what is it's relationship to the potential function ϕ?

(7) What mathematically is the difference between the waves in Figures 1 and 2?

(8) What in the article did you find interesting and valuable? What did you find confusing or frustrating?

Problem 7.B. You encountered a harmonic function in Socha's article. Here is a bit more on them. Before working on these problems, read the italicized text on harmonic functions on page 453.

(1) Do problems 28, 31, 32, and 33. Here are some hints and explanations.

For problem 28, recognize the path integral as one side of the equation that shows up in the 2D divergence theorem, so that the path integral can be thought of as the flux of a certain vector field across C. Figure out what that vector field is and apply the 2D divergence theorem.

For problem 32, you need to assume the function is harmonic. The point of these problems is that harmonic functions have the very special property that their values inside a region are determined by their values on the boundary. This is like saying that the value of the function where you are is determined by the values of the function at points 1000 miles from where you are – very strange! The problem is best done by being clear on the definition of normal derivative and recognizing that $\nabla f \cdot \nabla f = ||\nabla f||^2$. Combine this with the definition of $\partial f / \partial n$ (given in problem 30) and with exercise 31, to get your answer.

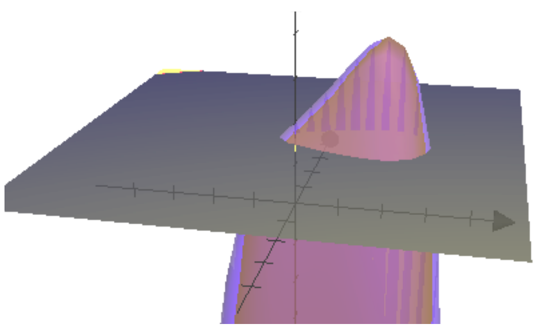
(2) Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a harmonic function. (This means that $\nabla^2 f = \operatorname{div} \nabla f = 0$.) Suppose that C is a closed contour (i.e. equipotential) line for f. (That is for all $(x,y) \in C$, $f(x,y)$ is constant.) Recall that $\mathbf{F} = \nabla f$ is always perpendicular to C. Use this fact and the 2D divergence theorem to show that:

$$\int_C ||\mathbf{F}||\, ds = 0.$$

(3) Explain why the previous part implies that if a harmonic function has a closed contour line then it is constant on the region bounded by the contour line.

(4) Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a harmonic function. Explain why there is no point $\mathbf{a} \in \mathbb{R}^2$ such that $\mathbf{a}$ is an isolated local maximum. (A point $\mathbf{a}$ is an isolated local maximum if, for all $\mathbf{x}$ close enough to $\mathbf{a}$ but not equal to $\mathbf{a}$, $f(\mathbf{x}) < f(\mathbf{a})$.)

Hint: Suppose that $\mathbf{a}$ was an isolated local maximum and use the picture below to argue that f must have a closed contour line enclosing $\mathbf{a}$. The picture below depicts the graph of a function with an isolated local maximum and a plane of constant height cutting through it.



Problem 7.C Read Section 7.1, up until the top of page 464 (before the discussion of area).

(1) Use Mathematica to plot the ellipsoid described by $x^2 + y^2 + 2z^2 = 1$ as a parameterized surface. (See the sample Mathematica file for how to do plot parametrically in 3 dimensions) You must do this parametrically and not implicitly. (Hint: Model your parametric equations on Example 2, but adjust them to get the ellipsoid instead.) Turn in a print out (in B&W) of your code and its output.

(2) Suppose that $\gamma\colon [a,b] \to \mathbb{R}^3$ and $\psi\colon [a,b] \to \mathbb{R}^3$ are smooth curves. Explain why the equation

$$\mathbf{F}(s,t) = (1-s)\gamma(t) + s\psi(t)$$

for $s \in [0,1]$ and $t \in [a,b]$ describes a parameterized surface made up of line segments joining the curves γ and ψ. What happens if we let $t \in \mathbb{R}$? Use this method to find a parameterization of the cylinder in Example 3 of Section 7.1 that is slightly different from the given one.

(3) Suppose that $Ax+By+Cz=D$ implicitly describes a plane in $\mathbb{R}^3$. Find a parameterization of it.

(4) Do problems 1, 2, and 14 from Section 7.1. (Hint for 14. Use part 2. above with a judicious choice of γ and ψ.)

Problem 7.D Read the rest of Section 7.1 and also 7.2.

Do problems 23, 34 from 7.1 and problems 1- 3, 6, 10 - 12 from Section 7.2. Remember to use Mathematica for computing integrals.