MA 262 Homework 6: Curl up and Div

Note: In addition to these problems, you should spend some time reviewing past material to study for the upcoming exam.

Problem 5.A Do these problems from the book. Remember that you may not consult solutions manuals, but you may ask me for help (in person or by email) and you may check your answers using Mathematica or the back of the book. These problems give you practice using Green's theorem.

Section 6.2 (pg 436 ff.): 1, 3, 5, 8, 10, 12, 30.

Problem 5.B Some Interesting Vector Fields

The relationship between vector fields and the region where they are defined is a very interesting one. It turns out that the topological type (eg. the number of holes) of the region has a big impact on the sort of vector fields that are defined on it. The next two problems provide some indication of the relationship. You may want to review HW 5 before doing this problem.

Let $\mathbf{F}_0(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$. Recall that in HW 5 you showed that $\mathbf{F}_0$ is not conservative but does have curl everywhere equal to zero. Notice also that $\mathbf{F}_0$ is defined everywhere except at the origin. Let $U = \mathbb{R}^2 \setminus \{(0,0)\}$

(1) Suppose that $\mathbf{G}$ is some other infinitely differentiable vector field on U with $\operatorname{scurl}\mathbf{G} = 0$ everywhere on U. Let C be a simple closed curve Use Green's theorem to explain why $\int_C \mathbf{G}\cdot d\mathbf{s} = 0$ if C does not enclose the origin.

(2) Using the same $\mathbf{G}$ as in (1). Let C_1 and C_2 be two simple closed curves both enclosing the origin and oriented counter-clockwise. You may assume that they don't intersect each other and that C_1 is to the inside of C_2. Use Green's theorem on the ring in between them to prove that $\int\limits_{C_1} \mathbf{G}\cdot d\mathbf{s} = \int\limits_{C_2} \mathbf{G}\cdot d\mathbf{s}$. Be sure to think about the role that the orientations of C_1 and C_2 play!

(3) Suppose that $\mathbf{G}$ is any infinitely differentiable vector field on U, as above and that $\operatorname{scurl}\mathbf{G} = 0$ everywhere on U. Let C be a simple closed curve oriented counter-clockwise and enclosing the origin. Suppose also that somehow we know that $\int_C \mathbf{G}\cdot d\mathbf{s} = \int_C \mathbf{F}_0\cdot d\mathbf{s}$. Explain why the vector field $\mathbf{F}_0 - \mathbf{G}$ is conservative. (Hint: Show that for any simple closed curve C' in U, $\int_{C'}(\mathbf{F}_0 - \mathbf{G})\cdot d\mathbf{s} = 0$. Appeal to a result from class and/or the reading.)

(4) Show that if $\mathbf{G}$ is any infinitely differentiable vector field on U with $\operatorname{scurl}\mathbf{G} = 0$ everywher on U, then there exists a real number k and a potential function $f\colon U \to \mathbb{R}$ such that

$$\mathbf{G} = k\mathbf{F}_0 + \nabla f.$$

In other words, up to scalar multiplication and addition of conservative vector fields, the vector field $\mathbf{F}_0$ is the unique vector field on U having scalar curl equal to 0. In fancy-pants language, $\mathbf{F}_0$ is a basis for the quotient vector space $\ker \operatorname{scurl} / \operatorname{im} \operatorname{grad}$ defined on U. This vector space is, therefore, 1–dimensional and the 1-dimension corresponds exactly to the one hole in U. This is part of a mathematical theory called "cohomology."

Problem 5.C (extra-credit!) Let U be the result of removing the points $(0,0)$ and $(1,0)$ from $\mathbb{R}^2$. Find two vector fields $\mathbf{G}_0$ and $\mathbf{G}_1$ that are infinitely differentiable on U, both have scalar curl equal to 0 on U and have the property that if $\mathbf{G}$ is any other such vector field on U, then there exist real numbers k, ℓ and a potential function $f\colon U \to \mathbb{R}$ such that

$$\mathbf{G} = k\mathbf{G}_0 + \ell\mathbf{G}_1 + \nabla f.$$

Be sure to explain why your vector fields $\mathbf{G}_0$ and $\mathbf{G}_1$ have all of the required properties! **Also** Explain why there are no $m \in \mathbb{R}$ and potential function $f\colon U \to \mathbb{R}$ so that $\mathbf{G}_1 = m\mathbf{G}_0 + \nabla f$.

This shows that the vector space $\ker \operatorname{scurl} / \operatorname{im} \operatorname{grad}$ defined on U is 2-dimensional. (Lo and behold! U has two holes!)