MA 262 Homework 5: Finding your path.

We have now entired the core content of the course: Vector Fields! Our goal for the next week is to develop an understanding of vector fields by studying paths moving through them.

Read Sections 3.4, 6.2 and 6.3. As you read focus on:

- The meaning of gradient, curl, and divergence.
- The integral and differential formulas for gradient, curl, and divergence.
- The relationship between gradient, curl and divergence.
- The special properties of conservative vector fields.
- The statement and meaning of Green's theorem.

Problem 5.A You might want to review the sample Mathematica file on the webpage for help with this one. Let $f(x,y) = x^2 + y$. Create a single Mathematica plot showing f (using DensityPlot), ∇f, and a flow line for ∇f. (Hint: first be sure you can plot each individually, then assign each of them a name and use the Show command. Get help from me early if you get stuck.) Turn in your Mathematica code along with what it produces. You can print it in black and white.

Problem 5.B Do these problems, each of which is just practice performing calculations. Recall that you may check your answers in the back of the book, but you should not make use of the solutions manual. Give credit on the cover sheet to any other sources you rely on. Feel free to use Mathematica to actually compute integrals.

Section 3.4 (pg 235 ff.): 1, 3, 5, 7, 8, 9, 10

Section 6.3 (pg 448 ff): 1, 2, 3, 4, 7

Section 6.2 (pg 436 ff): 1, 3, 5, 7.

Problem 5.C Recall that in class we defined the scalar curl of a vector field $\mathbf{F}$ on $\mathbb{R}^2$ by the integral formula:

$$\operatorname{scurl}\mathbf{F}(\mathbf{a}) = \lim_{n\to\infty} \frac{1}{\operatorname{area}(C_n)} \int_{C_n} \mathbf{F}\cdot d\mathbf{s}$$

where (C_n) is a sequence of closed curves (preferably rectangles) converging to $\mathbf{a}$ and $\operatorname{area}(C_n)$ is the area enclosed by each of them. Use this definition in conjunction with the Fundamental Theorem of Calculus for Conservative Vector Fields to prove that if $\mathbf{F}$ is a conservative vector field then its scalar curl at each point is zero.

Problem 5.D This is an important problem that we'll refer to repeatedly. The point is that whether or not a vector field is conservative depends not only on the formula defining $\mathbf{F}$ but also on the domain on which it is defined.

Let $\mathbf{F}(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$. This vector field is defined on the domain $U = \mathbb{R}^2 \setminus \{(0,0)\}$ (the plane minus the origin).

(1) Find a closed flow line and explain why this shows that $\mathbf{F}$ is not conservative on U.

(2) Let C be a circle of radius $r > 0$ centered at the origin, oriented counter-clockwise. Compute $\int_C \mathbf{F} \cdot d\mathbf{s}$. Explain why this computation also shows that $\mathbf{F}$ is not conservative on U.

(3) Find the scalar curl of $\mathbf{F}$ using the partial derivative formula. (The computation is a little messy, but the answer is pretty.)

(4) Let $P \in \mathbb{R}$. Find a vector field $\mathbf{G}$ on U such that $\operatorname{scurl} \mathbf{G} = 0$ and $\int_C \mathbf{G} \cdot d\mathbf{s} = P$. (Your answer will involve the number P and should be related to the vector field $\mathbf{F}$). Show that if $f\colon U \to \mathbb{R}$ is C^1, then $G + \nabla f$ also has this property.

(5) (You might want to save this one and the next for later in the week.) Suppose that C is any smooth simple closed curve not enclosing the origin. Use Green's theorem to show that $\int\limits_C \mathbf{F} \cdot d\mathbf{s} = 0$. Conclude that on any domain W not including the origin, $\mathbf{F}$ has path-independent line integrals and is, therefore, conservative.

Problem 5.E This problem looks much scarier than it actually is. We begin with some exposition:

A subset U of $\mathbb{R}^2$ is a **domain** if it has the property that whenever a point $\mathbf{x} \in U$ then all points sufficiently close to $\mathbf{x}$ are also elements of $\mathbb{R}^2$. For example, the set of points $(x,y) \in \mathbb{R}^2$ such that $(x,y) \neq (0,0)$ is a domain because if you take a point other than the origin and perturb it by a very small amount, it will still not be the origin. You'll be safe if you just think of a domain as a 2-dimensional subset of the plane. A line segment in $\mathbb{R}^2$, for instance, is not a domain.

A **vector space** (informally) is a set where we can add the elements of the set and also scale each element by a real number (called a "scalar".) For example, if X is any nonempty set, the set of functions $\mathscr{F}(X) = \{f\colon X \to \mathbb{R}\}$ is a vector space. For example, if f and g are both real-valued functions on X, we define the function $f+g$ by the formula:

$$(f+g)(x) = f(x) + g(x)$$

for every $x \in X$. Similarly, if $k \in \mathbb{R}$, we define the function kf by the formula:

$$(kf)(x) = k \cdot f(x)$$

for every $x \in X$. (Yes, this is really as simple as it looks. The point is that kf is the name of the new function while $kf(x)$ is what the function produces, when x is plugged in.)

(1) Let U be a domain in $\mathbb{R}^2$. Explain why the set of all infinitely differentiable scalar fields on U is a vector space. (This means you have to explain how to add two scalar fields to get another scalar field and how to scale a scalar field by a real number k.)

(2) Let U be a domain in $\mathbb{R}^2$. Explain why the set of all infinitely differentiable vector fields on U is a vector space.

Here is some more exposition: If V and W are vector spaces, a function $T\colon V \to W$ is a **linear transformation** if for every $a, b \in V$ we have $T(a+b) = T(a) + T(b)$ and for every $k \in \mathbb{R}$ we have $T(ka) = kT(a)$. For example, the function

$$T(x,y) = (2x + 3y, -x + 7y)$$

is a linear transformation from the vector space $\mathbb{R}^2$ to itself.

(3) (Warm-up 1) Let V be the vector space of all infinitely differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$. Explain why the derivative $\frac{d}{dt}\colon V \to V$ is a linear transformation. (You may just appeal to facts from Calculus 1 about how derivatives interact with addition of functions and multiplying by constants.)

(4) (Warm-up 2) Let V be the vector space of all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$. For $f \in V$, define

$$I(f) = \int_0^t f(\tau)\, d\tau.$$

The fundamental theorem of Calculus guarantees that $I(f)$ is a continuous (indeed, differentiable!) function. Use facts from Calculus 1 to explain why $I\colon V \to V$ is a linear transformation.

(5) Let $T\colon V \to W$ be a linear transformation between vector spaces V and W. Define the **kernel** of T to be the set:

$$\ker T = \{v \in V : T(v) = 0\}.$$

Define the **image** (or **range**) of T to be the set:

$$\operatorname{im} T = \{w \in W : \text{ there exists some } v \in V \text{ with } T(v) = w\}.$$

Show that $\ker T$ and $\operatorname{im} T$ are both vector spaces.

(Hint: To show that $\ker T$ is a vector space, suppose that $a \in \ker T$ and $b \in \ker T$. Explain why $T(a+b) = 0$, for then we see that $a + b \in \ker T$. Then suppose that $k \in \mathbb{R}$ and explain why $ka \in \ker T$. Finally, do a similar sort of thing for $\operatorname{im} T$.)

(6) Let U be a domain in $\mathbb{R}^2$ and let $C^0(U)$ be the vector space of all infinitely differentiable scalar fields on U. Let $C^1(U)$ denote the vector space of all infinitely differentiable vector fields on U. Explain why the gradient

$$\nabla\colon C^0(U) \to C^1(U)$$

is a linear transformation.

(7) Continue using the notation from the previous part. Recall that if $\mathbf{F} \in C^1(U)$, then we defined the **scalar curl** of $\mathbf{F}$ using the formula in 5.C above. Use that formula, along with properties of integrals and limits, to show that

$$\operatorname{scurl}\colon C^1(U) \to C^0(U)$$

is a linear transformation.

(8) We can also define scalar curl of a vector field $\mathbf{F}(x,y) = \begin{pmatrix} M(x,y) \\ N(x,y) \end{pmatrix}$ by the differential formula:

$$\operatorname{scurl}\mathbf{F}(\mathbf{x}) = \frac{\partial N}{\partial x}(\mathbf{x}) - \frac{\partial M}{\partial y}(\mathbf{x})$$

Use this version of the scalar curl to again prove that scalar curl is a linear transformation.

(9) Explain how divergence can be thought of as a linear transformation from $C^1(U) \to C^0(U)$.

(10) In class we discussed how for any scalar field f,

$$\operatorname{scurl}(\nabla f) = 0$$

Explain why this shows that every vector field that is in the image of the gradient is also in the kernel of scalar curl. (We write this as: $\operatorname{im}\nabla \subset \ker\operatorname{scurl}$.) Use 5.D. to give an example where the containment is proper; that is, there exists a vector field whose scalar curl is zero but which is not in the image of gradient.

Problem 5.F This problem expands on 5.D. Figure out how to adapt that problem to this context.

(1) Let $\mathbf{a} \in \mathbb{R}^2$. Let $U_{\mathbf{a}}$ consist of all of the plane except $\mathbf{a}$. Find a non-zero C^1 vector field $\mathbf{F_a}$ on $U_{\mathbf{a}}$ so that the scalar curl of $\mathbf{F_a}$ is zero but $\mathbf{F_a}$ is not conservative. Be sure to give a thorough explanation of how you know both facts.

(2) Do part (1) again, but his time suppose you are given a number $P \neq 0$ that, in addition to $\mathbf{F}$ having scalar curl 0 and being non-conservative, also ensure that if $C_{\mathbf{a}}$ is a circle enclosing $\mathbf{a}$, then

$$\int_{C_{\mathbf{a}}} \mathbf{F} \cdot d\mathbf{s} = P.$$

(Your answer will involve the number P.)

(3) Use 5.D to show that if C is any simple closed curve not enclosing $\mathbf{a}$, then $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$.

(4) Given distinct points $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}_2$, define $U_{\mathbf{a}}$ and $U_{\mathbf{b}}$ as above. Denote all the points of $\mathbb{R}^2$ that are neither $\mathbf{a}$ nor $\mathbf{b}$ by $U_{\mathbf{a}} \cap U_{\mathbf{b}}$. Let $C_{\mathbf{a}}$ be a small circle around the point $\mathbf{a}$ and $C_{\mathbf{b}}$ a small circle around $\mathbf{b}$. (Each should have radius smaller than $||\mathbf{b}-\mathbf{a}||$.) Find an example of a vector field $\mathbf{G}$ on $U_{\mathbf{a}} \cap U_{\mathbf{b}}$ such that $\operatorname{scurl}\mathbf{G} = 0$, $\int_{C_{\mathbf{a}}} \mathbf{G} \cdot d\mathbf{s} = 21$ and $\int_{C_{\mathbf{b}}} \mathbf{G} \cdot d\mathbf{s} = 9$.

Problem 5.G Let

$$\mathbf{F}(x,y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

for fixed values of a,b,c,d.

(1) Find the divergence of $\mathbf{F}$ using the partial derivative formula.

(2) Find the curl of $\mathbf{F}$ using the partial derivative formula.

(Those of you who have taken linear algebra may recognized that the divergence of a linear vector field is the trace.)