**Spring 2019/MA 262** **HW 4: Integrate your life!**

Read Sections 3.3 and 6.1

**4.A** From Section 3.2 (pp 219 and following) do problems 14, 15, and 17,19 (except you don't have to compute torsion). You may check your answers in the back of the book, but you may not consult the solutions manual.

**4.B** If $\mathbf{x}\colon [a,b] \to \mathbb{R}^2$ is a $C^1$ curve with non-zero speed, we define the **total curvature** of $\mathbf{x}$ to be the integral

$$\int_a^b \|\mathbf{T}'(t)\|\,dt.$$

In class we computed the total curvature of a circle (of any radius) and found it to be equal to $2\pi$.

(1) Let $r > 0$ and $k > 0$ be constants. Show that the total curvature of the path $\mathbf{x}(t) = (r\cos t, r\sin t, kt)$ for $0 \le t \le 2\pi$ is $2\pi r/\sqrt{r^2+k^2}$. What happens to the total curvature of the helix as $k$ increases or decreases? Make a connection to the total curvature of the circle.

(2) Prove that total curvature does not depend on the parameterization of the curve. That is, if $h\colon [c,d] \to [a,b]$ is a change of coordinates function then the total curvature of $\mathbf{y}(t) = \mathbf{x}(h(t))$ is the same as the total curvature of $\mathbf{x}$.

Hint: You need to use the chain rule when calculating $\mathbf{y}'$. When you integrate, you'll need to use substitution.

Remark: A theorem of Fenchel states that the total curvature of any closed curve in the plane is at least $2\pi$. At the age of 18, John Milnor proved that the total curvature of any knotted closed curve in $\mathbb{R}^3$ is at least $4\pi$. This theorem was independently proved by Fáry at about the same time.

**4.C** In the following, set up the problem first as a path integral (you may need to start by parameterizing the path) and then convert the path integral to a Calc I style integral. Then use NIntegrate in Mathematica to find a numerical approximation to your answer.

(1) A wire is in an oven which (with respect to a particular coordinate system) is heated so that the temperature at point $(x,y,z)$ is $(x^2+y^2)z$. The wire is a straight line formed by the intersection of the planes with equations $x+y+z=1$ and $2x-y+3z=0$ for $0 \le z \le 2$. Find the average temperature of the wire. (Hint: to parameterize the path find the endpoints and use a standard parameterization for a straight line.)

(2) An amusement park is surrounded by a fence that can be described by the ellipse $2x^2+5y^2=10$. The height of the fence at point $(x,y)$ is $(y+3)e^x$. Find the area of the fence.

(3) A field is covered with snow of varying depths. Using an appropriate coordinate system, it turns out that the depth of snow is described by $h(x,y) = \sin(xy)+2$. A child is dragging very thin stick beside him as he walks through the field so that the stick follows the path $(t\cos(t), t\sin(t)$ for $0 \le t \le 8\pi$. Find the total amount of area of snow moved by the stick.

(4) Problems 2 - 5 in Section 6.1.

**4.D** From Section 3.3, do:

(1) problems 1 - 6 by hand (sketching at least 5 arrows in each quadrant). Then check your answer using Mathematica. You may draw all arrows of the same length. Pay attention to how the different equations affect the vector field.

(2) problem 20

**4.E** From Section 6.1 do:

(1) Problems 8 - 15.

(2) Problem 25. If you don't know the legend of Sisyphus, look it up. My favorite pop culture reference to Sisyphus is in the movie "Party Girl".

**4.F** Give two explanations of each of the following facts. One explanation should be completely mathematical and the other intuitive and physics-based.

(1) If $f\colon \mathbb{R}^n \to \mathbb{R}$ is $C^1$ and if $\phi\colon [a,b] \to \mathbb{R}^n$ is a $C^1$ path, then $\int_\phi \nabla f \cdot d\mathbf{s} = f(\phi(b)) - f(\phi(a))$. (In other words, the path integral of a gradient field is just the difference of function values of the endpoints.

(2) If $\phi\colon [a,b] \to \mathbb{R}^n$ is a flow line for the vector field $\nabla f$ where $f\colon \mathbb{R}^n \to \mathbb{R}$ is $C^1$, then the integral $\int\limits_\phi \nabla f \cdot d\mathbf{s}$ is non-negative.

(3) Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}$ is a $C^1$ function and that $\mathbf{c}\colon [a,b] \to \mathbb{R}^n$ is a path such that there is a constant $k$ with $f(\mathbf{c}(t)) = k$ for all $t$. (That is, $\mathbf{c}$ is a **contour line** for $f$.) Then $\int_\mathbf{c} \nabla f \cdot d\mathbf{s} = 0$.