Spring 2019/MA 262 HW 2: Parameterized curves

The ultimate goal of this course is to understand the relationship between vector fields, scalar fields, parameterized curves, and parameterized surfaces. The goal of this homework assignment is to get you comfortable with parameterized curves. In a future homework assignment, we'll elaborate on some of the problems below.

Task 0: Read Sections 3.1 (you may skip the optional section) and 3.2. We'll cover 3.2 in a fair amount of detail in class. As you read pay attention to:

- The definition of velocity, speed, and acceleration of a path.
- The definition of length of a path. Why is it a sensible definition?
- How to reparameterize by arc length. This is a concept many students find challenging – take the time to absorb it.
- The definition of unit tangent vector and curvature

Problem 1: These problems are intended to give you experience calculating quantities of interest to us. You should also take the time to plot the paths using a computer so you can make connections between the formulas and the pictures. Remember: you may not use the solutions to the text and you must credit any source you rely on.

Part A: Section 3.1: 1- 3, 9, 25, 26

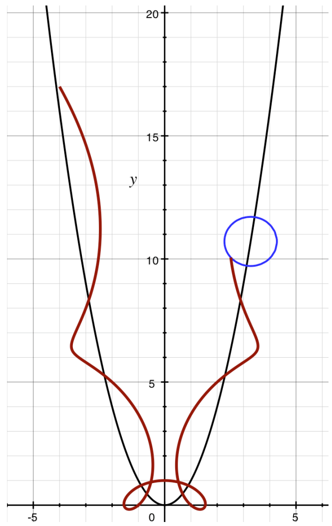
(Hint for problem 25: When the rocket's engines cease, all forces stop acting on the rocket. By Newton's 2nd law of motion this means that the acceleration of the rocket is 0 and so the rocket will follow the line tangent to the path it was on when the engines failed. Your task is to figure out if this path will bring it into the space station. You might also ask yourself whether, if you were on the space station, you'd actually want the rocket to arrive!)

(Hint for problem 26: The angle formed by the paths of the balls can be computed by using a dot product and the tangent vectors.)

Part B: Section 3.2: 1, 3, 8, 17. Use Mathematica to compute the integrals. You do not need to turn in a printout though. Just say you used Mathematica as part of your work on the problem.

Problem 2: Suppose that $\gamma\colon [a,b] \to \mathbb{R}^n$ (for $n = 2,3$) is a smooth path such that $||\gamma(t)|| = 1$ for every $t \in [a,b]$. (This means that the path lies on a circle if $n = 2$ or on a sphere if $n = 3$.) Use the dot product to show that at every $t \in [a,b]$, the tangent vector $\dot{\gamma}(t)$ is orthogonal to the vector $\gamma(t)$.

Problem 3.A Suppose that a circle of radius 1 is rolling down a hill such that the center of the circle is always on the graph of the parabola $y = x^2$. The circle rolls in such that the center of the circle is at the point (t, t^2) at time t and it completes 1 clockwise rotation every 2 seconds. At time $t = 0$, the center of the circle is at the point $(0,0)$. Let P be the point on the circle directly above the center of the circle at time $t = 0$. **Find** the parameterization of the path $\mathbf{x}(t)$ taken by the point P as the circle rolls down the parabola. See the image below for a depiction. **Use** Mathematica to plot the path you find (you do not have to plot the parabola and circle unless you want to.) **Turn** in a printout of your Mathematica code and picture with this problem.



Problem 3.B *The ride*

One of the most famous rides at Disney Land/World is the Mad Tea Party. Consider a version of this ride which consists of a large disc A that rotates counter-clockwise (viewed from above). Inside that disc and tangent to it are four smaller discs all (nearly) tangent to each other. These smaller discs rotate counter-clockwise (viewed from above, not accounting for the motion of A). Let B be one of these smaller discs. Inside B and tangent to it are four smaller circles (the tea cups), all (nearly) tangent to each other. These smaller circles spin at a variable rate determined by the people inside the teacup. Let C be one of them. Let P be a point on C (representing the position of someone on the ride). See the figure[1].

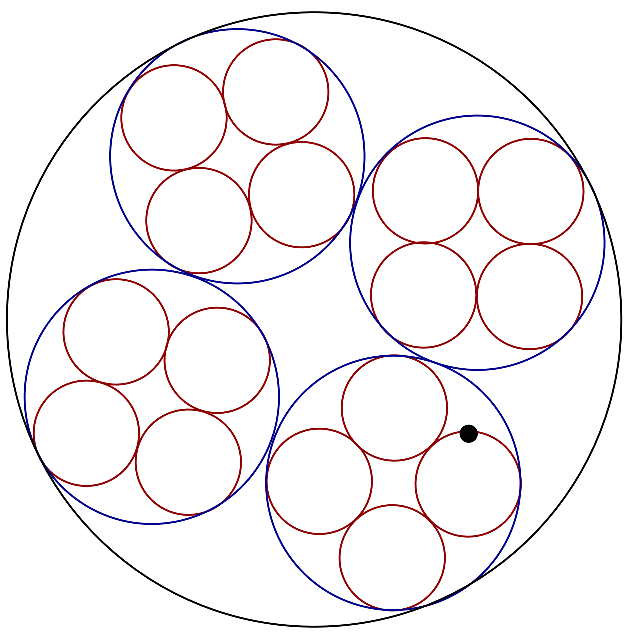


FIGURE 1. The point P is marked with a black dot

[1] In the actual ride, there are only 3 discs of Type B and they are not tangent and there are 5 teacups (not tangent) inside each of those discs.

Measurements

Suppose that units are chosen so that the radius of A is 1. Then the radius of B is $r_b = \frac{1}{1+\sqrt{2}}$ and the radius of C is $r_c = \frac{1}{(1+\sqrt{2})^2}$.

Suppose that A rotates at 1 revolution per second and that B rotates at 2 revolutions per second. The circle C rotates at $r(t)$ rotations per second, where if $r(t) < 0$, then C is rotating clockwise.

The centers

Choose coordinates so that the center of A is at the origin. Let $\mathbf{b}(t)$ denote the center of B at time t and let $\mathbf{c}(t)$ denote the center of C at time t. Let $\mathbf{P}(t)$ denote the position of P at time t. We assume that at time $t = 0$, the circles A, B, and C are all tangent to each other at the point $(1,0) = \mathbf{P}(0)$.

The problems

(1) (Extra-credit) Use elementary geometry to determine the radii of B and C given in the measurements section above.

(2) Find the coordinates of $\mathbf{b}(t)$. (Hint: The center of B always lies on a circle that is concentric with A and it moves at the same angular rate (and in the same direction) as A.)

(3) Find the coordinates of $\mathbf{c}(t)$. (Hint: The center of C always lies on a circle that is concentric with B and it moves at the same angular rate (and in the same direction) as B.)

(4) In tangent space coordinates based at $\mathbf{c}(t)$, find the coordinates of $\mathbf{P}(t)$. (Your answer will depend on $r(t)$, but not on any of the previous problems.)

(5) Find the coordinates of $\mathbf{P}(t)$ in the usual coordinate system. Your answer will, of course, depend on $r(t)$. **Use** Mathematica to plot the path you find (you do not have to plot the circles unless you want to.) **Turn** in a printout of your Mathematica code and picture with this problem.

(6) It turns out that if $r(t) = r$ is a constant, at time $t = 0$, we have:

$$||\mathbf{P}'(0)||^2 = \frac{4\pi^2(r + 3\sqrt{2} + 2)^2}{(1+\sqrt{2})^4}$$

Suppose that you are riding on the teacups with a 3-year old who is scared to travel too fast. How fast (in revolutions per second) and in what direction should you make the teacup rotate so as to minimize your speed at time $t = 0$? (You may restrict yourself to the situation where $r(t)$ is a constant independent of t.) Be sure to explain how you got your answer.