Actions and Permutations

1. Functions and Power sets

Suppose that $f\colon X \to Y$ is a function. This function induces a function $\widehat{f}\colon \mathscr{P}(X) \to \mathscr{P}(Y)$ on power sets, as follows. For each subset $A \subset X$, we define

$$\widehat{f}(A) = \text{range } f\Big|_A.$$

That is,

$$\widehat{f}(A) = \left\{b \in A : \exists\, a \in A \text{ s.t. } f(a) = b\right\} = \{f(a) : a \in A\}$$

Lemma 1.1. If f is a bijection, so is $\widehat{f}$.

Proof. This is easiest done by observing:

$$\widehat{f^{-1}} = (\widehat{f})^{-1}.$$

□

Lemma 1.2. Suppose that $f\colon X \to Y$ and $g\colon Y \to Z$ are functions, then

$$\widehat{g \circ f} = \widehat{g} \circ \widehat{f}.$$

Proof. We show that the functions on both the right and left take identical values on an arbitrary $A \in \mathscr{P}(X)$.

Let $x \in \widehat{g \circ f}(A)$. By definition, there is $a \in A$ such that $g \circ f(a) = x$. Let$b = f(a)$. Thus, $b \in \widehat{f}(A)$. Now $g(b) = x$. Thus, $x \in \widehat{g}(\widehat{f}(A))$. Consequently,

$$\widehat{g \circ f}(A) \subset \widehat{g} \circ \widehat{f}(A).$$

The claim that

$$\widehat{g} \circ \widehat{f}(A) \subset \widehat{g \circ f}(A),$$

is proved similarly. □

Notice that for every $A \in \mathscr{P}(X)$, the cardinality of $\widehat{f}(A)$ is the same as for A. Often times, it is advantageous to restrict $\widehat{f}$ to sets of a certain cardinality.

Perhaps confusingly, except in the next section, we *always* write $f(A)$ instead of $\widehat{f}(A)$. That is, we use the same notation for f and for $\widehat{f}$.

2. Permutations of subsets

Suppose that X is a set, possibly with some sort of structure (eg. a metric space or a group). Let $f\colon X \to X$ be a symmetry. Then we have the induced function $\widehat{f}\colon \mathscr{P}(X) \to \mathscr{P}(X)$ defined previously. By the definition of symmetry, f is a bijection and so $\widehat{f}$ is too. Thus, $\widehat{f} \in \textsc{Perm}(\mathscr{P}(X))$.

In fact, this produces an injective group homomorphism

$$\widehat{\cdot}\colon \text{Sym}(X) \to \textsc{Perm}\mathscr{P}(X).$$

It is defined by $f \mapsto \widehat{f}$.

Proof. Let $f, g \in \text{Sym}(X)$. Then $\widehat{f}, \widehat{g} \in \text{PERM}(\mathscr{P}(X))$. In the lemma of the last section we showed

$$\widehat{f \circ g} = \widehat{f} \circ \widehat{g}.$$

Since function composition is the group operation for both $\text{Sym}(X)$ and $\text{PERM}(\mathscr{P}(X))$, we have a group homomorphism.

To prove it's injective, simply notice that if $\widehat{f}(A) = A$ for every $A \subset X$, then $\widehat{f}(\{a\}) = \{a\}$ for every $a \in A$. Hence, $f(a) = a$ for every $a \in A$. Thus, f is the identity on X. □

Corollary 2.1. Let P be a property possibly possessed by some subsets of X and which is invariant under elements of $\text{Sym}(X)$ (i.e. P is part of the structure of X). Suppose that $f \in \text{Sym}(X)$, and that for some $k \geq 1$, $\mathscr{C}_k$ is the set of all subsets of X having cardinality k and some property P. Then there is a homomorphism $\text{Sym}(X) \to \text{PERM}(\mathscr{C}_k)$.

Proof. If A has k elements and if f is a bijection, then $\widehat{f}(A)$ also has k elements. □

Restricting to subsets of a fixed cardinality, the homomorphism of the previous corollary may not be injective. Anything in its kernel, fixes every k element subset having property P. But it may still move the points in those subsets around within the subsets.

Example 2.2. Let Γ be a graph and suppose $f \in \text{Sym}(\Gamma)$. Recall that f takes vertices to vertices and edges to edges. Thus, if A is a subset of the vertices of X such that no two distinct vertices of A are the endpoints of an edge in Γ, then $\widehat{f}(A)$ is a subset of the vertices of Γ such that no two vertices in $\widehat{f}(A)$ are the endpoints of an edge in Γ. Furthermore, $\widehat{f}(A)$ and A have the same cardinality. Suppose $\mathscr{C}_4$ is the set whose elements are the 4 element subsets A of the vertices of Γ such that no two vertices in A share an edge. We then have a homomorphism $\text{Sym}(\Gamma) \to \text{PERM}(\mathscr{C}_4)$.