Spring 2018/MA 434 HW 9: The group D_∞

1. HOMEWORK:

The group D_∞ has a lot of interesting properties. From the reading you know that it is infinite, non-abelian, and has a proper subgroup isomorphic to itself. In the exercise below, you'll show that it has a subgroup isomorphic to itself of every possible finite index and will show that its automorphism group is isomorphic to itself but is not equal to its inner automorphism group, which is also isomorphic to itself.

(1) Do problem (3) on page 52. Here are some hints.

  (a) Model your proof of the isomorphism claim on Prop. 2.2. For the claim about index, use Proposition 2.1 part 3 combined with Theorem 1.58. (Or model it on the end of the proof of Prop. 2.2)

  (b) Follow the pattern indicated by Prop 2.2 and part (a). You do not necessarily need to repeat the complete proof, just indicate what changes are needed to move from 2, to 3, to n.

  (c) Recall that an automorphism of D_∞ need not have a *geometric* meaning. It just is a bijective homomorphism of the group to itself. Suppose that α is such an automorphism. Assume that $\alpha(a)$ is a product of a and b of one of the forms in Prop. 2.1. Use the fact that automorphisms preserve orders of elements, to conclude that $\alpha(a)$ is a reflection of $\mathbb{R}$ across some $x = n$. Do a similar analysis to conclude that $\alpha(b)$ is a reflection of $\mathbb{R}$ across some $x = m$. Use Theorem 1.58 to argue that the subgroup generated by ab is of index 2 in D_∞ and thus the subgroup generated by $\alpha(ab)$ must also be of index 2. Use this in conjunction with Theorem 1.58 to conclude that $m = \pm 1$. (There may be other ways of doing this as well.)

  (d) To show that the center is trivial, recall that anything in the center must commute with a. Use Proposition 1.9. Also recall that for any group G, there is a natural homomorphism $G \to \mathrm{INN}(G)$. Its kernel is the center of the group.

  (e) Consider the automorphism ϕ given in the problem. Use Proposition 1.9 to argue that $\phi \notin \mathrm{INN}(G)$. By part (c), if α is an automorphism then $\alpha(a)$ is a reflection across $x = n$ for some integer n and b is a reflection across $x = n \pm 1$. If it is $n+1$, explain why $\alpha \in \mathrm{INN}(G)$. If it is $n-1$, show that $\alpha \circ \phi(a)$ is the reflection across $x = n-1$ and $\alpha \circ \phi(b)$ is the reflection across $x = n$. Conclude that $\alpha \circ \phi \in \mathrm{INN}(G)$. Thus, $\mathrm{INN}(G)$ must be of index 2.

  (f) Let ϕ be the automorphism from the previous part. Let $\psi\colon D_\infty \to D_\infty$ be conjugation by b. Use the previous part to prove that $\mathrm{AUT}(D_\infty)$ is generated by ϕ and ψ. Construct a homomorphism $f\colon D_\infty \to \mathrm{AUT}(D_\infty)$ by defining $f(a) = \phi$ and $f(b) = \psi$. Prove that f is an isomorphism.

(2) Do problem (5) on page 52. This should be relatively straightforward. There are four natural generators for the group W. Collect them in pairs. Show that each pair generates a subgroup isomorphic to D_∞ and that elements from different pairs commute with each other. (Hint: You can see this from the Cayley graph you construct!)