Spring 2018/MA 434 **Group Project 3: The Dodecahedron (and other stuff)**

1. Introduction

The dodecahedron X is the platonic solid consisting of 12 regular pentagons, arranged 3 around a vertex. See Figure 1.

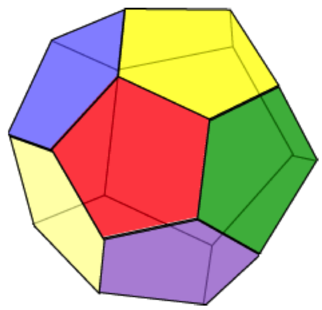

FIGURE 1. The dodecahedron. Image taken from paulscottinfo.ipage.com

Let I be the group of *rotational* symmetries of X. Since the dodecahedron and icosahedron are dual Platonic solids, they have isomorphic symmetry groups; I is called the icosahedral group.

Notice that I contains rotations by integer multiples of $2\pi/5$ around the center of each face of the dodecahedron, integer multiples of $2\pi/3$ around each vertex, and π around the center of each edge.

Answer the following questions:

(1) Prove that $|I| = 60$, preferably using the orbit-stabilizer theorem.

(2) Prove that I is vertex transitive, face-transitive, and edge-transitive.

(3) Consider a vertex v. What is $|\text{stab}(v)|$? If v and w are two distinct vertices, are there any elements in $\text{stab}(v) \cap \text{stab}(w)$? If so, what can you say about v and w?

(4) Prove that I contains at least 20 elements of order 3.

(5) Consider the center s of a face F. What is $|\text{stab}(s)|$? If s and s' are two distinct centers of faces, are there any elements in $\text{stab}(s) \cap \text{stab}(s')$? If so, what can you say about the faces?

(6) Prove that I contains at least 24 elements of order 5.

(7) Prove that I contains at least 15 elements of order 2.

(8) Observe that the previous lists of elements of I account for all the symmetries of G, apart from $\mathbb{1}$. In particular:

$$60 = 1 + 15 + 20 + 24.$$

Recall that conjugate elements have the same order. Prove that all the elements of order 2 are conjugate.

(9) Let v and w be distinct vertices of X. Let R be a rotation around v. Prove that R is conjugate to some rotation around w.

(10) Suppose v and w are *opposite* vertices of X. Let L be the line through them. For a rotation R around L, let $\theta(R,v)$ be the angle of rotation in the plane orthogonal to L with v above the plane and w below the plane. Explain why $\theta(R,v)=\theta^2(R,w)$. (Hint: think about what happens to a clock when you look at it in a mirror.)

(11) Prove that for a rotation R around a vertex, R is conjugate to R^2. Conclude that all elements of order 3 in I are conjugate.

(12) Prove that the rotations of order 5 are partitioned into two conjugacy classes. (This requires showing that they fall into at most two conjugacy classes and then that they fall into at least two conjugacy classes. For the latter, use the fact that the order of a conjugacy class divides the order of the group.)

(13) Prove that the class equation for I is:

$$60 = 1+15+20+12+12.$$

(14) Prove that I is a **simple** group. In other words, that it has no normal non-trivial proper subgroup. (Use the class equation and results from the previous group project.)

(15) It is possible to inscribe a cube inside X as in Figure 2. In fact, there are 5 such cubes. Explain.

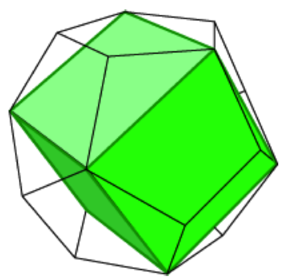

FIGURE 2. A cube inscribed in a dodecahedron. Image taken from paulscottinfo.ipage.com

(16) Number the cubes inscribed in X: $1,2,\ldots,5$. Construct an homomorphism $f: I \to S_5$ and prove it is injective. (Hint: Recall that some properties of the kernel of a homomorphism.)

(17) Recall that the **alternating group** A_5 is the kernel of the sign homomorphism $\epsilon: S_5 \to \{-1,+1\}$. (Recall that for a permutation σ, $\epsilon(\sigma)$ is $+1$ if σ is the composition of an even number of permutations, and -1 otherwise.) Prove that I is isomorphic to A_5.

(This project is based on material from Michael Artin's *Algebra*.)