Spring 2018/MA 434 **Group Project 2: Symmetries of Objects**

1. Introduction

In the previous group project, we saw how a group acts on itself via multiplication on the left. (This is sometimes called a **left translation**.) In the first part of this project, you will explore the action of a group on a geometric object and in the second part you'll explore a different kind of actions of G on itself.

2. Geometry

Recall (or learn!) that a **metric space** is a set X together with a way of measuring "distance". That is, a function $d\colon X \times X \to [0, \infty)$. The distance function (or **metric**) d is required to satisfy:

(M1) $d(x, y) = 0$ if and only if $x = y$.

(M2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(M3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We will use the following theorem without proof (though really every math major should learn how to prove it at some point.)

Theorem. *The set $\mathbb{R}^n$ with (euclidean) metric d defined by*

$$d(a, b) = \sqrt{(a-b)\cdot(a-b)}$$

is a metric space.

Here, the symbol $\cdot$ means the dot product. If we write a and b out in coordinates as $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$, then we can also write d as:

$$d\big((a_1, \ldots, a_n), (b_1, \ldots, b_n)\big) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2}.$$

Given any metric space a bijection $\phi\colon X \to X$ such that for all $a, b \in X$,

$$d(\phi(a), \phi(b)) = d(a, b)$$

is called an **isometry** of X. The set of isometries is denoted $\text{Isom}(X)$.

Here is another theorem whose proof we'll omit. (I usually cover it in *Geometry of Surfaces*.)

Theorem. *Every isometry T of $\mathbb{R}^n$ (with the euclidean metric) is of the form:*

$$T(x) = Ax + b$$

*where $b \in \mathbb{R}^n$ and A is an orthogonal $n \times n$ matrix. If $\det A > 0$ the isometry is **orientation-preserving**; otherwise it is **orientation-reversing**.*

In the case of 2 and 3-dimensions, the theorem takes the slighly nicer form.

Theorem. *Suppose that $T \in \text{Isom}(\mathbb{R}^2)$. Then T is the composition of translations, rotations, and reflections. If T is orientation-preserving, then it is the composition of a single translation and a single rotation.*

Theorem. *Suppose that $T \in \textsc{Isom}(\mathbb{R}^3)$. Then T is the composition of translations, rotations and reflections. If T is orientation-preserving, then it is the composition of a single translation and a single rotation.*

Proof. Suppose that $T \in \textsc{Isom}(\mathbb{R}^3)$ is orientation-preserving and that $T(0) = 0$. By the theorem above, there exists an orthogonal matrix A such that $T(x) = Ax$ for every $x \in \mathbb{R}^3$. The definition of orthogonal matrix says that $AA^T = \mathbb{1}$. The multiplicative property of the determinant says that $\det(A) = \pm 1$. Since it is orientation-preserving, $\det(A) = 1$. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues for A. Their product is equal to 1. Since complex eigenvalues come in conjugate pairs, there is (at least) one $\lambda \in \mathbb{R}$. Let v be the associated eigen-vector. We have

$$Av = \lambda v$$

It follows from the fact that A is orthogonal (or that T is an isometry) that $\lambda = \pm 1$. If $\lambda = -1$, then all three eigenvalues are real and there is a different one which is $+1$. Hence, we may assume that $\lambda = 1$. Writing A in the basis corresponding to unit length eigenvectors, we see that A is a rotation around the line with axis passing through 0 and v.

If $T(0) \neq 0$, then apply the previous paragraph to $T - T(0)$. If T is not orientation-preserving, apply it to the composition of T with a reflection. □

If $X \subset \mathbb{R}^n$ an **isometry** of X is the restriction of an isometry of $\mathbb{R}^n$ to X.

Answer the following questions:

(1) You were given an archimedean solid. Analyze, as best you can, the isometries of your object. For various points, work out their orbits and stabilizers. What are the orders of the different symmetries? Can you figure out how many symmetries your object has? How many orientation-preserving symmetries does it have? For a polyhedron $X \subset \mathbb{R}^n$, a group acting by isometries on X is said to be **vertex-transitive** if all of the vertices are in a single orbit. Is the isometry group of your object vertex-transitive?

Write a paragraph summarizing your findings.

(2) Let R be a rotation of $\mathbb{R}^2$ by an angle θ around the origin. Let S be a rotation by an angle θ around a point $p \in \mathbb{R}^2$ other than the origin. Prove that R and S are conjugate in $\textsc{Isom}(\mathbb{R}^2)$. That is, show that there exists $T \in \textsc{Isom}(\mathbb{R}^2)$ such that

$$S = TRT^{-1}.$$

(3) Let R be a reflection across a line $L \subset \mathbb{R}^2$. Let S be a reflection across a line $M \subset \mathbb{R}^2$. Prove that R and S are conjugate in $\textsc{Isom}(\mathbb{R}^2)$.

(4) Prove that, in $\textsc{Isom}(\mathbb{R}^2)$, orientation-preserving isometries and orientation-reversing isometries are never conjugate.

(5) Letting X be your archimedean solid, find two conjugate isometries of X.

Moral: If f and g are conjugate elements of a group G acting on a set S, we can think of f and g as "doing the same thing," but in different locations.

3. THE INNER AUTOMORPHISM GROUP

Let G be a group and let $S = G$. Define a left action $G \curvearrowright S$ by

$$(g, s) \mapsto g s g^{-1}$$

It is called the action by **left conjugation**.

Answer the following questions:

(1) Show that the action by left conjugation of a group on itself is an action.

(2) Given $g \in G$, let $\phi_g \colon G \to G$ be defined by $\phi_g(s) = g s g^{-1}$. Show that ϕ_g is an automorphism (i.e. isomorphism) of G. It is called an **inner automorphism** of G.

(3) Show that the set $\textsc{Aut}(G)$ of all automorphisms of G is a group and that the set $\textsc{Inn}(G)$ of inner automorphisms of G is a normal subgroup. Recall that whenever we quotient by a normal subgroup, the resulting set of cosets inherits a group structure. In this case, the resulting group is called the **outer automorphism group** of G.

(4) Switching our gaze back to the action by left conjugation, define the **conjugacy class** of $s \in G$ to simply be its orbit under the action. That is, the conjugacy class $C(s)$ of s is the set of elements to which it is conjugate. Let $\mathscr{C} \subset G$ be set containing exactly one element from each conjugacy class in G. From our previous work, we know that orbits partition the set so (recalling that, in this case, G is the set). We have (when $|G|$ is finite) the **class formula**

$$|G| = \sum_{s \in \mathscr{C}} |C(s)|$$

Recalling that $C(\mathbb{1}) = \{\mathbb{1}\}$, we can rewrite this as:

$$|G| = 1 + \sum_{c\, s \in \mathscr{C}, s \neq \mathbb{1}} |C(s)|.$$

Since each conjugacy class $C(s)$ is an orbit under an action of G on itself, by the orbit stabilizer theorem $|C(s)|$ must divide $|G|$. As observed in Artin's *Algebra*, this puts very strong restrictions on either the size of the group or the size of the conjugacy classes in G.

Prove that $g \in G$ is in the center of G if and only if $|C(G)| = 1$. (The **center** of a group is the set of all elements of G which commute with all other elements of G.) Conclude that if $|G|$ is a power of a prime, then the center of G is non-trivial.

(5) Prove that if $|G| = p^2$ for some prime p, then G is abelian.

(6) Let D_n be the dihedral group of order $2n$. (Recall that D_n is the set of isometries of a regular n-gon in $\mathbb{R}^2$.) Determine all the conjugacy classes in D_n and verify that the class equation holds.

(7) Prove that if G is a group and if $H \triangleleft G$, then H is the union of (some of the) conjugacy classes in G. Conclude that (when $H \triangleleft G$)

$$|H| = 1 + \sum_{s \in \mathscr{C} \cap H, s \neq 1} |C(s)|$$