Spring 2018/MA 434 **Group Project 1: Orbit-Stabilizer Theorem**

1. Recollections

Recall that a **group** is a set G together with an operation $\circ$ such that:

(G1) For all $a, b \in G$, there is a unique element $a \circ b \in G$

(G2) There exists an element $\mathbb{1}$ such that for every $a \in G$,

$$a \circ \mathbb{1} = \mathbb{1} \circ a = a$$

(G3) For every $a \in G$, there exists $a^{-1} \in G$ such that

$$a \circ a^{-1} = a^{-1} \circ a = \mathbb{1}$$

(G4) For every $a, b, c \in G$

$$a \circ (b \circ c) = (a \circ b) \circ c$$

We often write ab instead of $a \circ b$. Recall that $\mathbb{1}$ is the unique element in G satisfying the equations in (G2) and that, given $a \in G$, the element a^{-1} is the unique element in G satisfying the equation in (G3). If a group G also satisfies, for every $a, b \in G$,

$$ab = ba$$

we say that G is **abelian**.

If $H \subset G$ is also a group (with the same operation $\circ$) then H is a **subgroup** of G. If $H \subset G$ is a subgroup, we write $H < G$. It is a **normal** subgroup if for every $h \in H$ and $g \in G$, $ghg^{-1} \in H$. In general, we say that the element ghg^{-1} is the **conjugate** of h by g. If $H < G$ is a normal subgroup, we write $H \triangleleft G$.

Given a group G and $H < G$, recall that for $g \in G$, a **left coset** of H is the set:

$$gH = \{g' \in G : \exists h \in H \text{ s.t. } g' = gh\}$$

The set $G/H = \{gH : g \in G\}$ is called the **quotient set** of G by H. The cardinal number $[G : H] = |G/H|$ is called the **index** of H in G. It can be infinite.

Given groups G and G', a **homomorphism** is a function $\phi\colon G \to G'$ such that for all $a, b \in G$

$$\phi(ab) = \phi(a)\phi(b).$$

It is an **isomorphism** if it is also a bijection.

Write answers to the following questions:

(1) List as many different groups or types of groups as you can. Be sure to include finite and infinite groups and abelian and non-abelian groups.

(2) Prove that G/H is a partition of G. (Equivalently, defining $g \sim g'$ if and only if there exists $h \in H$ with $g' = gh$ makes $\sim$ an equivalence relation on G.)

(3) Lagrange's theorem says that if G is a finite group and if $H < G$, then

$$|G| = |H|[G : H]$$

Prove it, using the fact that G/H is a partition of G and by showing that each coset gH is in bijection with H.

2. Group Actions

A **permutation** of a set X is a bijection $X \to X$. The set of permutations of X is denoted $\text{PERM}(X)$, it is a group with function composition as the operation.

Definition. A **left action** of a group G on a set S is a function $G \times S \to S$ such that, for every $g \in G$ and $s \in s$,

$$(g,s) \mapsto gs \in S$$

and

(A1) $\mathbb{1}s = s$ for every $s \in S$
(A2) $g(g's) = (gg')s$ for every $g, g' \in G$ and $s \in S$.

If G acts on S on the left, we write $G \curvearrowright S$, although this notation doesn't indicate what the action is. We usually refer to the elements of S as **points**.

Write answers to the following questions:

(1) Consider the set $S = \{1, \dots, n\}$ and the **symmetric group** $S_n = \text{PERM}(S)$. Explain how S_n acts on S (on the left).

(2) Let S be any set and G be any group. Define $gs = s$ for every $g \in G$ and $s \in S$. Prove that this is a left action. This is called the **trivial** action.

(3) Let $G = GL(2, \mathbb{R})$ be the group of invertible 2×2 matrices with real entries. Explain how G acts (non-trivially) on $\mathbb{R}^2$.

(4) Suppose that $G \curvearrowright S$ and that $s \in S$ is a **fixed point**. That is, a point $s \in S$ such that $gs = s$ for every $g \in G$. Prove that $G \curvearrowright S \setminus \{s\}$.

(5) Suppose that G acts on S on the left. For $g \in G$, let $\phi_g \colon S \to S$ be the function

$$\phi_g(s) = gs.$$

Prove that ϕ_g is a permutation of S and that $g \mapsto \phi_g$ is a homomorphism of G into $\text{PERM}(S)$.

(6) Conversely, suppose that $\phi \colon G \to \text{PERM}(S)$ is a homomorphism from a group G into the group of permutations of some set S. Prove that the map $(g,s) \mapsto \phi(g)(s)$ defines a left group action.

(7) Let G be a group with operation $\circ$ and let $S = G$. Define **left multiplication** by

$$(g,s) = g \circ s$$

Prove that (letting $gs = g \circ s$) this is a left action of G on itself.

(8) Put the previous parts together to prove Cayley's Theorem: Every group is isomorphic to a subgroup of a permutation group.

3. The orbit-stabilizer theorem

Suppose that G acts on S on the left. For a given $s \in S$, the **orbit** of s is:

$$\text{orb}(s) = \{s' \in S : \exists g \in G : s' = gs\}$$

Notice that $\text{orb}(s) \subset S$. It is the set of points to which s can move by applying elements of G. If we need to keep track of the group, we'll write $\text{orb}_G(s)$.

For $s \in S$, define the **stabilizer** of s to be

$$\mathrm{stab}(s) = \{g \in G : gs = s\}$$

Notice that $\mathrm{stab}(s) \subset G$. It is the set of elements of G which do not move the point s. If we need to keep track of the group, we'll write $\mathrm{stab}_G(s)$.

Write answers to the following. Assume throughout that $G \curvearrowright S$.

(1) Prove that for every $s \in S$, the set $\mathrm{stab}(s)$ is a subgroup of G.

(2) Suppose that $G \curvearrowright S$. Prove that $\{\mathrm{orb}(s) : s \in S\}$ is a partition of S.

(3) Let $s \in S$. Find a bijection between $G/\mathrm{stab}(S)$ and $\mathrm{orb}(s)$.

(4) Prove the **orbit-stabilizer** theorem: If G is a finite group acting on a set S, then for every $s \in S$:

$$|G| = |\mathrm{orb}(s)||\mathrm{stab}(s)|.$$

(5) Suppose that G is a finite group acting on a finite set S such that the action does not have any fixed point (i.e. $\nexists s \in S$ s.t. $\mathrm{stab}(s) = G$). Prove that if $|G|$ is a power of a prime then $|S|$ is a multiple of that prime.