Spring 2018/MA 434 **Group Project 5: Group actions on Graphs have fundamental domains**

Throughout, suppose that G is a group acting on a connected simple graph Γ. Assume, for simplicity, that every vertex of Γ has finite degree and that Γ has countably many vertices. We consider Γ as a metric space where every edge has length 1. Thus, it makes sense to talk about the midpoint of an edge.

We want to show that there is a subset $\mathscr{F} \subset \Gamma$ which has the following properties:

(1) It is closed (in the sense of metric spaces)
(2) Its translates cover Γ. That is, $\bigcup_{x \in \mathscr{F}} \text{orb}(x) = \Gamma$.
(3) No proper subset of $\mathscr{F}$ satisfies (1) and (2).

Our construction of $\mathscr{F}$ proceeds algorithmically.

For each $i \in \mathbb{N}$, we'll let S_i be a list (i.e. finite sequence) of vertices to consider adding to $\mathscr{F}$. We'll let F_i be the subgraph consisting of those vertices we've added to $\mathscr{F}$ along with all edges having both endpoints in the set. We'll define each F_i so that no two vertices are in the same orbit.

For any vertex w in Γ, let $N(w)$ be its neighbors in Γ.

I encourage you to draw a graph with a group action and follow along the following construction.

(1) Choose a vertex $v \in \Gamma$. Set $F_1 = \{v\}$. Number the neighbors of v as $v_1, v_2, \ldots, v_n$. Add them one at a time to the list S_1, but obey the following rule: A vertex v_i is added to S_1 as long as it is not in the orbit of any of the other vertices previously added to S_1 and is also not in the orbit of v.

(2) Suppose that w is a neighbor of v which is in the orbit of v. Suppose that w' is a neighbor of w which is *not* in the orbit of v. Prove that there is a neighbor v' of v which is not in the orbit of v such that w' is in the orbit of v'.

(3) Let F_2 be the subgraph consisting of the vertex v and v_1 and the edge between them. Remove v_1 from S_1. Look at the list of neighbors of v_1. Add them to the end of S_1 to obtain the list S_2, but obey the following rule: A vertex is added to the list as long as it is not in the orbit of any other vertex already in the list and is not in the orbit of any vertex in F_2.

(4) Verify that no vertices in F_1 or S_2 are in the same orbit.

(5) Suppose that we have defined subgraphs

$$F_1 \subset F_2 \subset \cdots \subset F_k$$

and lists

$$S_1, S_2, \ldots, S_k$$

as follows. For $2 \leq i \leq k$, the subgraph F_i is obtained from F_{i-1} by adding the first vertex w of S_i to F_{i-1} along with any edges having both endpoints in F_i. The set S_i is obtained from S_{i-1} by removing the first element of S_{i-1} and adding vertices to the end of the list. The vertices we add are those neighbors of w which are not in the orbit of any vertex of F_i or the vertices previously added to the list.

Prove that no two vertices of the subgraph F_i are in the same orbit. Explain why this means that no two edges of F_i are in the same orbit.

(6) Explain how to define F_{k+1} and S_{k+1} so that the sequence can be continued. What, if anything, might keep us from making this definition?

(7) By induction, we have a (possibly infinite) sequence of subgraphs

$$F_1 \subset F_2 \subset \cdots$$

Let $F = \bigcup F_k$ be their union.

(a) Prove that F is a subgraph of Γ. (That is, explain why ever edge of F has both endpoints in F.)
(b) Note that F is closed (all subgraphs are closed).
(c) Prove that no two vertices of F are in the same orbit.
(d) Prove that no two edges of F are in the same orbit.

(8) Prove that every vertex of Γ is in the orbit of some vertex of F.

(9) Consider the action of a cyclic group on a wheel with spokes. (For example $\oplus$.) Show that it may be the case that not every edge of G is in the orbit of F.

(10) Show how to add edges or half edges to F to obtain the desired set $\mathscr{F}$.

(This project is based on material from John Meier's *Groups, Graphs, and Trees.*)