MA 274: Exam 2 Study Guide (UPDATED 4/17)

Here are some suggestions for what and how to study:

Theorems marked with (X) will not be on the exam. Theorems marked (CHALLENGE) are particularly challenging and would placed on the exam only as a challenge problem.

(1) Know the definitions on the website. Any other definitions that you need will be given to you.

(2) When you write a proof, focus on getting the organization clear and correct. If you have to skip some steps or make an assumption that you don't know how to prove, clearly state that that is what you are doing.

(3) Know the theorems we've proved in class and the more significant theorems from the homework.

(4) Don't try to memorize proofs. Instead remember the structure of the proof (proof by contradiction, proof of uniqueness, element argument, etc.) and two or three key steps of the proof. Then at the exam recreate the proof.

(5) At the exam, leave time to write up a nicely written version of each proof. You should have enough time to sketch your ideas out on scratch paper before writing a final version of the proof.

(6) Here are some results you should be especially sure to know how to prove; some of them may be new. You should also think about ways these problems might be varied.

(7) **Use the examlet study guide too!**

   (a) The following are examples of equivalence relations:

   - $\equiv_p$ on $\mathbb{Z}$, where $x \equiv_p y$ iff $x - y$ is a multiple of p.
   - $\sim$ on the vertex set of a graph G, where $x \sim y$ iff there is a path from x to y in the graph.
   - Suppose that P is a partition of a non-empty set X. Define $\sim$ on X by $x \sim y$ iff x, y are both elements of the same element of P.

   (b) Suppose that $\sim$ is an equivalence relation on a non-empty set X. For each $x \in X$, let $[x]$ be the equivalence class of x. Then the following hold:

   (i) For all $x \in X$, $x \in [x]$.
   (ii) For all $x, y \in X$, $x \sim y$ iff $[x] = [y]$.
   (iii) For all $x, y \in X$, $[x] \cap [y] \neq \varnothing$ implies $[x] = [y]$.

   (c) If $\sim$ is an equivalence relation on a non-empty set X, then the set of equivalence classes is a partition of X.

(d) If $f\colon \mathbb{Z}/\equiv_p \to \mathbb{Z}/\equiv_p$ is defined by $f([x]) = [2x]$ then f is well-defined.

(e) Suppose that (x_n) is the sequence in S^1 defined by letting x_n be the result of rotating $x_0 = (1,0)$ by an angle of $n\theta$ for some fixed θ. Then (x_n) is periodic iff θ is a rational multiple of π.

(f) The compositions of injections/surjections/bijections is a an injection/surjection/bijection.

(g) A function $f\colon X \to Y$ is a bijection if and only if there is a function $f^{-1}\colon X \to Y$ such that $f \circ f^{-1}(y) = y$ for all $y \in Y$ and $f^{-1} \circ f(x) = x$ for all $x \in X$.

(h) The set of bijections from a set X to itself is a group, with function composition as the operation.

(i) Every element of $\mathbb{N}^*$ is either even or is one more than an even number.

(j) There are $n!$ permutations of an n-element set.

(k) For every permutation of the set $\{1,\dots,n\}$, the permutation can be written as the composition of transpositions.

(l) A convex polygon having $n \geq 3$ sides can be triangulated with $n-2$ triangles.

(m) A polygon having $n \geq 3$ sides can be triangulated with $n-2$ triangles. (You may use the fact that every polygon completely contains a line segment joining two of its vertices.)

(n) Every natural number $n \geq 2$ has a prime factorization. That is, there exists $m \in \mathbb{N}$ and prime numbers $p_1,\dots,p_m$ such that $n = p_1 p_2 \cdots p_m$.

(o) (X) The Well-Ordering Principle

(p) (CHALLENGE) For every $n, m \in \mathbb{N}$, if there is a bijection from $\{1,\dots,n\}$ to $\{1,\dots m\}$, then $n = m$.

(q) (X) If G is a graph and a, b are vertices of G such that there is a path from a to b then there is a shortest path from a to b.

(r) (X) If G is a connected graph such that a and b are vertices and e is some edge (not necessarily having a and b as endpoints) then there exists a path from a to b crossing e at most once.

(s) (X) If G is a connected graph and e is an edge then the graph obtained by removing e from G has at most two connected components.

(t) Euler's theorem for planar graphs: If G is a finite, planar, non-empty, connected graph with $V(G)$ vertices, $E(G)$ edges, and $F(G)$ faces, then $V(G) - E(G) + F(G) = 2$.

(u) (X) Every fraction can be written in lowest terms.

(v) (ADDED 4/17) Prove that if X and Y are non-empty sets then $X = Y$ if and only if $X \times Y = Y \times X$.