Spring 2018/MA 274 GP 1E: Modular Arithmetic

Consider the set of real numbers $\mathbb{R}$.

(1) For real numbers x and y, define the symbol $x \equiv y$ to mean that $x - y$ is an integer multiple of 2π. More precisely, $x \equiv y$ if and only if there is an integer $n \in \mathbb{Z}$ such that

$$x - y = 2\pi n.$$

Explain why the following are true:

(a) For every $x \in \mathbb{R}$, $x \equiv x$. (To show this, you need to specify what value of n makes it so that $x - x = 2\pi n$.)
(b) For every $x, y \in \mathbb{Z}$, if $x \equiv y$, then also $y \equiv x$. (You get to *assume* there is an n so that $x - y = 2\pi n$. You need to *show* that $y \equiv x$.)
(c) For every $x, y, z \in \mathbb{Z}$, if $x \equiv y$ and $y \equiv z$, then $x \equiv z$.

(2) List all (or list some and say what the pattern is) the real numbers y such that $0 \equiv y$.

(3) List all the integers y such that $\pi \equiv y$.

(4) List all the integers y such that $\pi/3 \equiv y$.

(5) Assume that $x \equiv y$ and that $a \equiv b$. Show that $(x + a) \equiv (y + b)$.

(6) For a real number $x \in \mathbb{R}$, let $[x]$ be the set of all real numbers y, such that $x \equiv y$. This is called the **equivalence class** of x. Note that you listed the elements of $[0]$, $[\pi]$, and $[\pi/3]$ above. Explain the following:

(a) If $x \equiv y$, then $[x] \subset [y]$.
(b) If $y \equiv x$, then $[y] \subset [x]$. By the previous work, this shows that if $x \equiv y$ then $[x] = [y]$.
(c) If $x \sim y$ and $a \sim b$, then $[x + a] = [y + b]$.

The point of the previous problems is that we can define addition on the sets $[x]$, by defining

$$[x] + [a] = [x + a].$$

On the left is the addition of sets which we don't know how to do. On the right is the addition of real numbers, which we know how to do. Since if $x \sim y$ and $a \sim b$, then $[x + a] = [y + b]$, we conclude that if $[x] = [y]$ and $[a] = [b]$, then $[x] + [a] = [y] + [b]$. This looks like an obvious kind of statement, but we don't know that it is true until we prove it. Just because we *want* $+$ and $=$ to behave a certain way, doesn't mean they do!

For your presentation, explain your answers to all these questions.