Read Section 2.3 up through Exercise 2.3.6.

(1) Do Exercise 2.3.5

(2) Figure 1 shows a way of moving through the graph along a path.

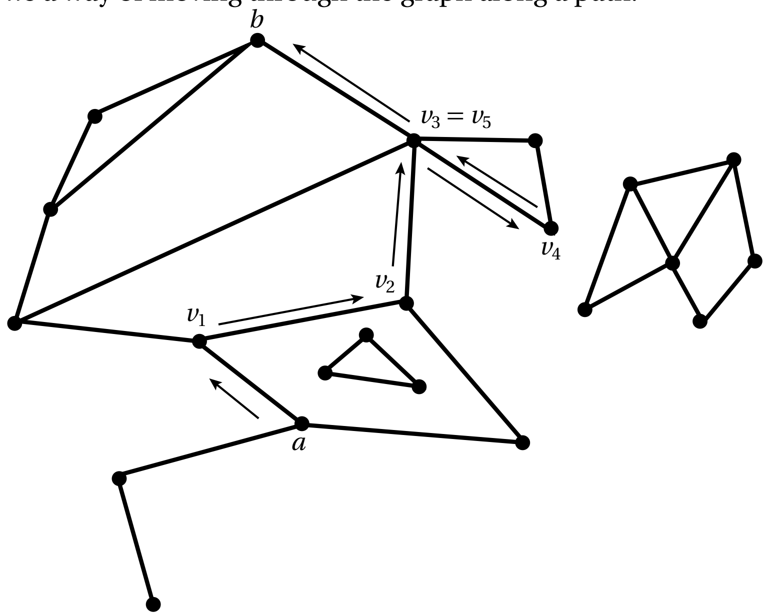

FIGURE 1. The finite sequence $a, v_1, v_2, v_3, v_4, v_5, b$ is a path from a to b of length 6.

To specify our route, we can list all the vertices we pass through:

$$a, v_1, v_2, v_3, v_4, v_5, b$$

The vertex v_3 is the same as the vertex v_5, so there are vertices that appear more than once in the list.

Draw your own example of a fairly complicated graph and specify a path in it between two vertices.

(3) Here is the formal definition of a path in a graph:

Definition. Suppose that $G = (V, E)$ is a graph . A **path** in G is a finite sequence

$$\alpha = v_0, v_1, \ldots, v_n$$

for some $n \in \mathbb{N}^*$ such that the following conditions hold:

- Each v_i (for $i \in \{0, \ldots, n\}$) is a vertex of G.
- For all $i \in \{0, \ldots, n-1\}$, the vertices v_i and v_{i+1} are the endpoints of an edge in G.

The number n is the **length** of α. If e is an edge of G such that there exists $i \in \{0, \ldots, n-1\}$ with the endpoints of e equal to v_i and v_{i+1}, then we say that e is **traversed** by α. If a and b are vertices of G such that $v_0 = a$ and $v_n = b$, we say that α is a path **from** a **to** b.

Explain how the formal definition applies to the example given above and to the example you created.

(4) A graph is **connected** if for every two vertices a and b, there is a path from a to b. For a connected graph G, and for vertices a and b define $d(a,b)$ to be the length of the shortest path from a to b. Prove the following:

   (a) For every vertex a, $d(a,a)=0$. (Hint: A path can consist of a single vertex.)
   (b) For all vertices a and b, we have $d(a,b)=d(b,a)$. (Hint: Suppose you have the shortest path from a to b. How can you get the shortest path from b to a?)
   (c) For all vertices a,b,c, we have $d(a,c)\leq d(a,b)+d(b,c)$. (Hint: Start with a shortest path α from a to b and a shortest path β from b to c. Explain how to construct some path from a to c whose length is $d(a,b)+d(b,c)$. Then explain why $d(a,c)$ is no more than that length.)

(5) Explain why a connected graph, together with the way of measuring distance d given in the previous problem, is a metric space.

For your presentation, explain the definition of path, the definition of length of a path and your answers to (4) and (5).