

Course Location: *Section A (10 AM)* in Mudd 405
Section B (12 n.) in Lovejoy 303

Professor: Scott Taylor

Email: scott.taylor@colby.edu

Webpage: <http://www.colby.edu/~sataylor>

Office Hours: *Tuesday* 2:15 – 3:15
(subject to change) *Wednesday* 2 – 2:45
Thursday 12 – 1:30
Friday 1 – 2
and by appointment!

Office: Davis 207

Prerequisites: MA 102, 121, 122, or 161 with 2 semesters of Calc recommended

Text: Taylor, *S. Number, Space, and the Structures of Mathematics*
available online. Password is: QED

The Course:

Modern mathematics is based on surprisingly few structures all of which have similar traits. This semester you'll learn how to communicate technical mathematics by proving theorems pertaining to these basic mathematical structures. Along the way, we'll see how these structures can be used to describe and explain a variety of phenomena from the arts and sciences.

Both the writing skills you develop and the mathematical ideas you'll encounter are essential for all future courses in pure mathematics and many courses in applied mathematics. More generally, the ability to devise, structure and effectively communicate using precisely defined concepts related in complex ways to the experienced world is crucial to many other technical disciplines (such as statistics, computer science, and law).

The best way to improve your ability to write and create mathematics is to do it. Thus, this course is structured so that you first encounter new concepts on your own, have the opportunity to wrestle with them, and then go to class where misunderstandings can be cleared up and gaps can be filled. This is certainly the best way to learn this material. At the end of this syllabus, you will find suggestions for how to succeed in this course – I encourage you to contemplate them frequently. We are in this together – you are always welcome to stop by my office with questions or comments. My task is to guide you on your mathematical journey, steer you clear from potholes, and help you keep the end in view.

Objectives for increasing mathematical maturity:

By the conclusion of the course students will have improved in their ability to:

- Use definitions, axioms, and previously proved results to arrive at correct proofs of theorems by careful logical reasoning.
- Write coherent proofs in a conventional mathematical style
- Prove theorems using direct proofs, proofs by contradiction, proofs by induction, and proofs by contraposition
- Be able to generate examples of or counterexamples to mathematical claims
- Give coherent written and verbal explanations of mathematics
- Evaluate the correctness or incorrectness of a proof
- Engage in significant self-teaching of mathematics
- Be comfortable when encountering new mathematical concepts
- Understand something of the goals, methods, and culture of several different mathematical disciplines
- Be able to express technical ideas in both formal and informal ways.

Specific Course Content Objectives:

The students will study in some detail: sets and functions, graphs, groups and metric spaces, equivalence relations, sequences, and cardinality. For each of these topics, students should be able to give precise definitions of the most important concepts and use them to prove theorems. Specifically, students should:

- be able to prove DeMorgan's laws and Russell's paradox
- be able to prove that certain properties persist under arbitrary intersections or unions.
- be able to calculate or bound the cardinalities of various finite and infinite sets
- be able to prove basic results from graph theory, group theory, and the theory of metric spaces
- be able to prove results concerning the injectivity and surjectivity of certain functions
- be able to prove that certain functions have certain properties (e.g. they are homomorphisms, isometries, or are continuous.)
- be able to prove that certain functions are "well-defined".
- be able to construct sequences and subsequences in metric spaces which have certain properties
- be able to prove that certain sequences converge or don't converge
- be able to prove that a given relation is or is not an equivalence relation
- be able to prove that certain properties persist from a set to the quotient set.
- be able to prove Euler's theorem on the number of vertices, edges, and faces of a planar graph.
- be able to prove LaGrange's theorem from group theory
- be able to prove that the image of a point under a rotations by an irrational multiple of π are dense in the unit circle.

Evaluation:

The numerical course grade will be a weighted average of the cumulative grades with weightings as follows:

- 15% examlets
- 20% Exam 2
- 25% Final exam
- 25% Daily/Weekly homework
- 10% Proofs for the Public Paper
- 5% Small group projects, LaTeX, and Class participation

however, any student receiving less than 50% on both Exam 2 and the Final Exam will receive an F in the course.

Course letter grades will be assigned (subject to above caveat) according to the following scale. Any curve will be determined at the end of the course, according to the discretion of the instructor.

93 - 100 %	A	73 - 77 %	C
90 - 93 %	A-	70 - 73 %	C-
87 - 90 %	B+	67 - 70 %	D+
83 - 87 %	B	63 - 67 %	D
80 - 83 %	B-	60 - 63 %	D-
77 - 80 %	C+	below 60 %	F

Students who demonstrate an exceptional interest and ability in the course may be given an A+.

Academic Honesty

Academic honesty means that the work you present is your own, the ideas you communicate actually represent what you think and feel, and that you are upfront about your sources and inspirations. An act of academic dishonesty can either be intentional or unintentional. In either case, there are both informal and formal consequences. Being scrupulously honest is essential for the functioning of the college, for getting the most out of your own education, and for the success of mathematics as a discipline.

In this course, the greatest temptation towards academic dishonesty is presenting someone else's proof as your own (perhaps with minor modifications). If someone other than the TA or professor gives you the key idea for a proof, you must credit them in your write-up. *This includes any help you receive from online sources and applies even if you received help from a student who received help from a TA or the professor.* Acknowledging a source will never result in penalty. You are encouraged to work together and share ideas, but you may not copy another person's proof. Furthermore, you may not share LaTeX source code with other students and, in your final paper, you must give complete citations for any ideas, phrasings, or images you get from another source.

Academic dishonesty is a serious offense against the college. Sanctions for academic dishonesty are assigned by an academic review board and may include failure on the assignment, failure in the course, or suspension or expulsion from the College.

Learning Differences:

Students with learning differences are encouraged to meet with me to discuss strategies for success. I am committed to helping all students succeed and to making reasonable accommodations for documented learning differences.

Exams:

Almost every Monday, beginning on Feb. 13, the end of the course period will be devoted to an examlet. Essentially, an examlet consists of one or two questions suitable for a midterm exam. Typically you will be asked to define a term and to prove a theorem previously encountered in the course. You should take these seriously and study for them. They can be rescheduled only in the case of documented illness, personal tragedy, or religious observance. Students with extended time accommodations should contact me at the beginning of the semester to make arrangements. *Up to 3% extra-credit can be earned on the cumulative examlet score by attending up to 3 math/stats department colloquia and writing a paragraph summarizing the talk and a paragraph responding to the talk.*

There will be one midterm exam on the evening of **Tuesday, April 25** from 7 – 9 PM in a location TBD. In the event that you are unable to attend the scheduled exam, you must let the instructor know *in advance*.

The final exam is during exam period **7** on **Friday, May 12** from **9 - 12**. It is college policy that the final exam cannot be rescheduled for personal convenience, *including airline reservations*. In the event that you have three or more final exams scheduled in close proximity, it is sometimes possible to reschedule one or more of them. See the registrar's webpage for details.

The examlets and two exams are cumulative, although the final exam is “more cumulative”. The two exams will present you with new mathematics and will expect you to prove theorems. One of the goals of the course is to equip you to succeed in such a situation. As with homework, your exam grade will be based not only on the correctness of your solutions, but also on your ability to clearly and coherently communicate them.

Small group projects, LaTeX, and class participation:

This course is designed to teach you to write, discuss, and evaluate new mathematics. You cannot learn to do this unless you are actively involved and participating. It is expected that you will be present in class and will make substantial contributions to class discussion. Since everyone is required to participate it is expected that you will allow your classmates the opportunity to do so and that all classroom interactions will be conducted with decorum and respect.

You will engage in several small group projects over the course of the semester. These will typically involve discussing and proving a theorem together in groups of 2 or 3. These groups will be assigned. Each person in the group will individually be responsible for typing (in the mathematical typesetting system LaTeX) both a proof of the theorem and a summary of the process by which the group came to find the solution. The solutions, the writing, and the process will be graded. It is possible that not everyone in the group will receive the same grade. A member of the group may be asked to present the solution at the board to the class.

Final “Proofs for the Public” paper:

The ability to communicate to both technically adept and technically unsophisticated audiences is essential for those working in mathematical disciplines. The final paper gives you the opportunity to translate technical ideas into a form which can be understood and appreciated by non-technical audiences. More details will be distributed later, but it is worth pointing out here that you will need to use the mathematical typesetting system LaTeX to type your paper. Again, details will be forthcoming.

Daily/Weekly Homework:

Homework will typically consist of both reading and writing. The reading will be from the textbook, notes distributed on the website, and other articles provided throughout the semester. Your writing will consist mostly of written proofs but will also contain responses and analyses to the articles.

Daily homework will be assigned and will be collected weekly. ***However, you must have completed the assigned reading prior to class after which it was assigned.*** All homework will eventually be placed in your journal (see below) and it is expected that whatever you turn in will follow the formatting requirements for the journal (see below). **Most every week after the first, at least one proof must be typed using LaTeX. You will be told which solution should be typed.**

You may work with other people on the homework assignments, but the write-up must be your own. I suggest that you find a study group and meet regularly with them to discuss the homework. *Then*, go off by yourself and write up your solutions yourself.

Typically, a daily assignment will consist of reading several sections from the text, solving all or most of the exercises and problems and providing proofs for some of the theorems in the chapter. Each week you will turn in the daily assignments from the previous week. Assignments should be submitted by placing them in the “official homework binders”. Late assignments may be penalized; please tell me if you are having trouble meeting a deadline.

Note: if you wait until just before you are to turn in the daily assignments to attempt them, you will not succeed and you will not be prepared for class.

In each class meeting, I will assume that you have done the previous daily homework and can discuss it intelligently.

After the graded assignment is returned to you, some problems may be marked with (*). The presence (*) means your solution met a certain threshold of acceptability and that you may redo the problem to earn back any missing points. The acceptable threshold will be determined by the professor or grader and will not be communicated in advance. The threshold will be based on the quality of the writing and the quality of the mathematics – you should take the problem seriously the first time through it. A rewrite of a problem must be turned in with the next assignment.

Formatting for homework:

- At the top of each page should be written:
 - Your name
 - Assignment Number
 - The due date of the assignment
 - The part of the assignment that the contents of the page fall under. The required parts of each assignment will be clearly indicated when each assignment is given.

For example:

Scott Taylor
Assignment #2

September 10
Definitions from Sections 1.1 – 1.4

or

Scott Taylor
Assignment #2

September 10
Part 1 (4a): Exercise 1.5.2

- No double-sided page should contain content from more than one part of an assignment or from more than one assignment or should contain the proof of more than one theorem. (Closely related results with short proofs may, however, be on the same page.)
- All assignments, parts of assignments, and exercises must be in the correct order. See below for more details.
- No page should contain prominent erasure marks, cross-outs, writing in the margin, etc. All writing must be completely legible. If you need to rewrite an entire page before submitting your work – do so! Particularly for proofs of theorems, the first draft of a proof is **not** the one to submit.
- For each assigned exercise or problem, write the exercise or problem number and the statement of the problem if it is brief. Leave a blank line or two and write your solution. If a problem has multiple parts and the solution to one or more parts takes up a significant amount of space, you may write the statement of each part above the solution for that part, rather than grouping all the statements together. You have permission to be somewhat flexible in the precise formatting of your writing. You should, however, always aim for your work to be so readable that the grader can read and understand what you have written without referring to the text.

See the following guidelines for writing proofs. You must write in complete sentences. You may find it helpful to pretend that you are writing a solutions manual that will be read by students who are not as capable as yourself.

Writing Proofs

The strong proof will exhibit *careful logical reasoning* combined with *succinct expression* written with the inexorable drive of compelling mathematical ideas. A proof exhibits “careful logical reasoning” when each statement follows directly and immediately from a definition, axiom, or previously proved result, or a small number of these combined with one or more rules of logic. A proof exhibits “succinct expression” when illustrative examples are not used in place of careful logical reasoning, logical steps considered ‘obvious’ by the reader and which are not necessary, are omitted, and connecting words and transitions between sections while present are as brief and to the point as possible.

Guidelines for a successful assignment write-up¹:

1. Your target audience is your classmates, NOT me or the grader! Your classmates should be able to read and understand your solution without having to ask you any questions about it.
2. You should use only full sentences, mathematical or otherwise. When using mathematical symbols convert them to words in your head to make sure that you are not missing appropriate verbs, articles, etc. Remember that mathematical language has a more rigid format (not unlike a programming language) than written English.
3. Writing many vague or repetitive sentences cannot be a substitute for saying the right thing once. Think your ideas all the way through before you shape them into steps of an argument. **Preliminary drafts are strongly recommended!**
4. I suggest placing consequent “steps” of an argument on separate lines, similar to the way this is done in instruction manuals, or in presentations of computer algorithms. At the very least, make good use of paragraph breaks.
5. Display longer formulas and diagrams on separate lines the way this is done in textbooks.
6. Skipping steps of an argument or asserting that something is “obvious” (henceforth called a “magic leap”) can be considered a significant or even fatal error. I and the grader are the sole judges of what constitutes a magic leap or an insignificant error, so stay on the safe side and remember part 1.!
7. Do not put the proofs of two different theorems on the same piece of double-sided paper, unless both proofs are short and you are certain they will not need to be rewritten.
8. Your write-up should be laid out carefully and presented in a neat manner. Do a rough draft first, since a sloppy presentation will be penalized. The harder and less pleasant it is for us to follow your argument, the higher are the chances of you receiving a lower grade for the write-up.

¹ Adapted from guidelines used by Leo Livshits.

For example:

1.7.4 Example

Construct converses of the following statements:

1. If Elsie is a cow, then Elsie is a mammal.
2. If $x = 0$, then $x^2 = 0$.

Solution:

1. If Elsie is a mammal, then Elsie is a cow.
2. If $x^2 = 0$, then $x = 0$.

Here is an example of how to format the proof of a theorem:

8.2.2 Theorem (Uniqueness of Inverses)

Let x be a real number. There exists a unique real number y such that $x + y = 0$. Similarly, if x is not zero, there exists a unique y such that $x \cdot y = 1$.

Proof:

Claim 1: If x is a real number then there exists a unique real number y such that $x + y = 0$.

Proof of Claim 1: We must prove both that y exists and that it is unique. By Axiom 1 (pg 180), there exists an additive inverse y for x . That is, there exists y so that $x + y = 0$.

It remains to show that y is unique. To that end, suppose that y and z are both inverses for x . By the definition of additive inverse:

$$(a.) \quad \begin{aligned} x + y &= 0, \text{ and} \\ x + z &= 0. \end{aligned}$$

Since each thing is equal to itself:

$$y = y.$$

Since 0 is the additive identity (Axiom 1), $0 + y = y$.

Consequently,

$$0 + y = 0 + y.$$

Thus, by equations (a.) we have:

$$(y + x) + y = (z + x) + y.$$

Since addition is associative (Axiom 1),

$$y + (x + y) = z + (x + y).$$

Consequently, by equation (a.),

$$y + 0 = z + 0.$$

Since 0 is the additive identity (Axiom 1),

$$y = z.$$

Hence, additive inverses are unique as desired. Q.E.D (Claim 1)

Claim 2: If x is a non-zero real number, there exists a unique real number y such that $x \cdot y = 1$.

Proof of Claim 2: We must prove both that y exists and that it is unique. By Axiom 1 (pg 180), there exists a multiplicative inverse y for x . That is, there exists y so that $x \cdot y = 1$.

It remains to show that y is unique. To that end, suppose that y and z are both inverses for x . By the definition of multiplicative inverse:

$$(a.) \quad \begin{aligned} x \cdot y &= 1, \text{ and} \\ x \cdot z &= 1. \end{aligned}$$

Since each thing is equal to itself:

$$y = y.$$

Since 1 is the multiplicative identity (Axiom 1), $1 \cdot y = y$.

Consequently,

$$1 \cdot y = 1 \cdot y.$$

Thus, by equations (a.) we have:

$$(y \cdot x) \cdot y = (z \cdot x) \cdot y.$$

Since multiplication is associative (Axiom 1),

$$y \cdot (x \cdot y) = z \cdot (x \cdot y).$$

Consequently, by equation (a.),

$$y \cdot 0 = z \cdot 0.$$

Since 0 is the multiplicative identity (Axiom 1):

$$y = z.$$

Hence, multiplicative inverses are unique as desired.

Q.E.D (Claim 2.)

Since we have proven both Claims 1 and 2, Q.E.D.

Other Sources:

- Pages 1 – 10 of these notes from a class by Donald Knuth:
<http://tex.loria.fr/typographie/mathwriting.pdf>
- Sections 13-16 of this classic article by Paul Halmos:
http://golem.ph.utexas.edu/category/2009/10/halmos_on_writing_mathematics.html
(Follow the obvious link to the article.)
- Chapter 0 of *Mathematical Proofs* by Chartrand, Polimeni, and Zhang.

Sexual Misconduct/Title IX Statement:

Colby College prohibits and will not tolerate sexual misconduct or gender-based discrimination of any kind. Colby is legally obligated to investigate sexual misconduct (including, but not limited to sexual assault and sexual harassment).

If you wish to speak confidentially about an incident of sexual misconduct, please contact Colby Counseling Services ([207-859-4490](tel:207-859-4490)) or the Director of the Gender and Sexual Diversity Program, Emily Schusterbauer ([207-859-4093](tel:207-859-4093)).

Students should be aware that faculty members are considered responsible employees; as such, if you disclose an incident of sexual misconduct to a faculty member, they have an obligation to report it to Colby's Title IX Coordinator. "Disclosure" may include communication in-person, via email/phone/text, or through class assignments.

To learn more about sexual misconduct or report an incident, visit <http://www.colby.edu/sexualviolence/>.

How to succeed in this course:

MA 274 is an intense course and likely differs considerably from other math classes that you have taken. I recommend the following to help you do your very best in the course and to get the most out of it.

1. Do some math everyday. Really.
Learning mathematics is a lot like learning to play an instrument, play a sport, or learn a language. You must practice everyday. The homework is intended to help you in your daily practice. But you should spend additional time each day reading the text and studying previous material. On average, you should spend 2 - 3 hours studying for each hour spent in class. In our case, that's at least 6 – 9 hours per week of homework and studying.
2. Participate in class. Yes, you.
Asking and answering questions is a great way to stay engaged with the material and verify for yourself that you know what's going on. The more people that participate, the more fun class is. I, and the rest of the class, value your questions

and your answers, right or wrong. In fact, giving a wrong answer to a question is a great way to learn the right answer. Make an effort to connect each class's activities to previous classes. Try to predict where the material will be going in the future. Take good notes and listen. If you can't do both, make a deal with a buddy: you take notes and they listen one day and the next day you switch. And, finally, volunteer to present in class. This is a great way to get feedback on your writing and proving.

3. Read the textbook.

The lectures and the textbook will often present slightly different views on the same material. You will usually read the textbook before discussing the material in class. Think about the difference between the approaches and how they inform each other. After class, reread the section and see how your understanding has changed.

4. Form a study group.

Introduce yourself to other people in the class and meet up outside of class to study and work on homework. The fourth floor of Mudd is a great place to meet in the evenings. Be sure that you don't copy answers, but learn from each other and then write the answers on your own. Compare lecture notes to be sure you copied everything correctly. Ask each other questions and explain material to each other.

5. Write your own problems.

As you study for the exams, look back at all the examples done in class, in the text, or on homework. Try modifying them to make up your own problems. Try to solve your own problems – what makes your problems easier or more difficult than the ones you've seen before? Feel free to show me what you've done.

6. Be curious.

Ask lots of questions. Ask questions to connect our course with material from other courses – especially writing, philosophy, and mathematics courses. Try to predict where the course is going. Revisit material from previous math classes and figure out how it could have been presented if you had taken MA 274 first. Try to create and prove your own theorems. Be curious – this is one of the privileges of being in college.

7. Visit me.

I love working with students and I love to help you understand and appreciate the beautiful world of mathematics. Feel free to drop by, even when it's not my office hours. If I can't chat, I'll let you know. Ask crazy questions about the course. Ask questions about my research. Tell me about your past math experiences. Tell me about what subjects you love. Let me know when you have a concert or athletic event. Colby has a take-your-professor to lunch program where I get free lunch if you sign me in. Let's have lunch!

8. Spread the studying out over the semester.

If you do math everyday, as suggested above, you won't have to cram for exams. You'll be able to sleep and, consequently, to think. You'll be happier and more relaxed. You'll have time to write papers for your other classes. You'll have time to appreciate the New England winter. You don't need to pull all-nighters.

9. Have an exam strategy.

For the midterm exam, you will have only 2 hours in which you have to do some significant mathematics. Be prepared to do some problems very rapidly and be prepared to think about others. If an example or proof was done in class or on homework, you should be able to repeat it very quickly on the exam. Know what you find difficult and what you find easy. Do the easy things first and then the difficult things. Write something for every problem. If you get stuck, tell me how you'd solve it if you could get unstuck. Figure out what the problem is testing and tell me what you know about that area. If the problem is too hard, rewrite it to make it easier. I love to give partial credit. Give me a reason to give you some. If you find yourself getting nervous: breathe deeply, remind yourself you've studied thoroughly, then figure out how to do the problem. Keep an eye on the time and don't spend too long on any one problem.