S17 MA 274: Exam 3 Study Questions

You can find solutions to some of these problems on the next page. These questions only pertain to material covered since Exam 2. The final exam is cumulative, so you should also study earlier material.

(1) Know the definitions on the website. Any other definitions that you need will be given to you.

(2) When you write a proof, focus on getting the organization clear and correct. If you have to skip some steps or make an assumption that you don't know how to prove, clearly state that that is what you are doing.

(3) Know the theorems we've proved in class and the more significant theorems from the homework.

(4) Don't try to memorize proofs. Instead remember the structure of the proof (proof by contradiction, proof of uniqueness, element argument, etc.) and two or three key steps of the proof. Then at the exam recreate the proof.

(5) At the exam, leave time to write up a nicely written version of each proof. You should have enough time to sketch your ideas out on scratch paper before writing a final version of the proof.

(6) Study the previous study guides and exams as well as your homework, class notes, and the sections of the text we covered.

Prove the following:

(1) The number $\sqrt{2}$ is irrational.

(2) There are infinitely many prime numbers.

(3) There is no set U such that $A \in U$ if and only if A is a set. (Russell's Paradox)

(4) The Halting Problem

(5) DeMorgan's Laws

(6) Suppose G is a group with operation $\circ$ and that $a \in G$. If $f, g \in G$ have the properties that $f \circ a = a \circ f = a$ and $g \circ a = a \circ g = a$, then $f = g$. (That is, the identity in a group is unique.)

(7) Suppose that G is a graph and that a, b, and c are vertices. Then if there is a path from a to b and a path from b to c, then there is a path from a to c. (See HW 7 for the definitions.)

(8) The intersection of subgroups is a subgroup

(9) The intersection of convex sets is convex

(10) The intersection of event spaces is an event space.

(11) $X \times Y = Y \times X$ if and only if either $X = Y$ or one of X or Y is empty.

(12) If (x_n) is a sequence in a set X such that for all $N \in \mathbb{N}$ there exists $n \geq N$ such that $x_n \notin \{x_1, \ldots, x_N\}$ then (x_n) has a subsequence (x_{n_k}) which is injective and such that $\text{range}(x_{n_k}) = \text{range}(x_n)$.

(13) Suppose that (x_n) is a sequence in $\mathbb{R}$ with the property that for all $N \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $x_m < \min\{x_1, \dots, x_N\}$. Prove that (x_n) has a subsequence (x_{n_k}) which is strictly decreasing.

(14) Be able to prove that something is or is not an equivalence relation. For example, if $\equiv_7$ is defined on $\mathbb{Z}$ by $x \equiv_7 y$ if and only if $x - y$ is a multiple of 7, prove that $\equiv_7$ is an equivalence relation.

(15) Be able to prove that if $\sim$ is an equivalence relation on X and if f is a given function with domain $X/\!\sim$ then f is well-defined. For example, if we define $f\colon\ \mathbb{Z}/\!\sim\, \to \mathbb{Z}/\!\sim$ by $f([x]) = [2x]$ then f is well-defined.

(16) Addition of equivalence classes in $\mathbb{Z}/\!\equiv_p$ is well-defined. That is, Define $x \equiv_p y$ if and only $x - y$ is a multiple of p. Define $[x] + [y] = [x + y]$. Prove that $[x] + [y]$ is well-defined.

(17) Addition on $\mathbb{Q}^+ = (\mathbb{N} \times \mathbb{N})/\!\sim$ is well-defined where pairs $(x, y) \sim (a, b)$ if and only if $xb = ya$.

(18) If $\sim$ is an equivalence relation on X, then for all $x, y \in X$, if $[x] \cap [y] \neq \varnothing$ then $[x] = [y]$.

(19) If $\sim$ is an equivalence relation on X, then $x \sim y$ if and only if $[x] = [y]$.

(20) If $\sim$ is an equivalence relation on X, then $X/\!\sim$ is a partition of X.

(21) State and prove LaGrange's theorem.

(22) Let S^1 be the unit circle. For any $\alpha \in \mathbb{R}$, let R_α be the counterclockwise rotation by α radians. (If $\alpha < 0$ this means rotate by $|\alpha|$ radians clockwise.) Suppose that $\theta \in \mathbb{R}$. Let (x_n) be the sequence in S^1 where $x_0 = (1, 0)$ and $x_n = R_\theta(x_{n-1})$ for all $n \in \mathbb{N}$. Prove the following:

  (a) The sequence (x_n) is injective if and only if $\theta \notin \pi\mathbb{Q}$ (i.e. θ is not a rational multiple of π.)
  (b) The sequence (x_n) is periodic (i.e. there exists $n \in \mathbb{N}$ such that $x_n = x_0$) if and only if θ is a rational multiple of π.
  (c) The sequence (x_n) is not surjective.
  (d) If $\theta \notin \pi\mathbb{Q}$, then there exists a subsequence (x_{n_k}) converging to x_0.

(23) (a new one) Let $X = \mathcal{P}(\mathbb{R})$ and define $\sim$ on X by $A \sim B$ if and only if there exists a bijection $f\colon\ A \to B$. Prove that $\sim$ is an equivalence relation.

(24) (a new one) Let X be a non-empty set and let $\mathcal{F}$ be the set of bijections of X to itself (i.e. permutations of X). For $f, g \in \mathcal{F}$ define $f \sim g$ if and only if there exists a bijection $h \in \mathcal{F}$ such that

$$f = h^{-1} \circ g \circ h.$$

Prove that $\sim$ is an equivalence relation.