MA 274: Exam 2 Study Guide

Here are some suggestions for what and how to study:

(1) Know the definitions on the website. Any other definitions that you need will be given to you.

(2) When you write a proof, focus on getting the organization clear and correct. If you have to skip some steps or make an assumption that you don't know how to prove, clearly state that that is what you are doing.

(3) Know the theorems we've proved in class and the more significant theorems from the homework.

(4) Don't try to memorize proofs. Instead remember the structure of the proof (proof by contradiction, proof of uniqueness, element argument, etc.) and two or three key steps of the proof. Then at the exam recreate the proof.

(5) At the exam, leave time to write up a nicely written version of each proof. You should have enough time to sketch your ideas out on scratch paper before writing a final version of the proof.

(6) Here are some results you should be especially sure to know how to prove. You should also think about ways these problems might be varied.

(7) **Use the examlet study guide too!**

   (a) The compositions of injections/surjections/bijections is a an injection/surjection/bijection.

   (b) A function $f\colon X \to Y$ is a bijection if and only if there is a function $f^{-1}\colon X \to Y$ such that $f \circ f^{-1}(y) = y$ for all $y \in Y$ and $f^{-1} \circ f(x) = x$ for all $x \in X$.

   (c) A convex polygon having $n \geq 3$ sides can be triangulated with $n-2$ triangles.

   (d) A polygon having $n \geq 3$ sides can be triangulated with $n-2$ triangles. (You may use the fact that every polygon completely contains a line segment joining two of its vertices.)

   (e) Every natural number $n \geq 2$ has a prime factorization. That is, there exists $m \in \mathbb{N}$ and prime numbers $p_1, \ldots, p_m$ such that $n = p_1 p_2 \cdots p_m$.

   (f) The Well-Ordering Principle

   (g) For every $n, m \in \mathbb{N}$, if there is a bijection from $\{1, \ldots, n\}$ to $\{1, \ldots m\}$, then $n = m$.

   (h) If there are bijections $X \to \{1, \ldots, n\}$ and $X \to \{1, \ldots, m\}$ for some $n, m \in \mathbb{N}$. Then $n = m$. (Be careful: this is harder than it looks!)

(i) If G is a graph and a, b are vertices of G such that there is a path from a to b then there is a shortest path from a to b.

(j) If G is a connected graph such that a and b are vertices and e is some edge (not necessarily having a and b as endpoints) then there exists a path from a to b crossing e at most once.

(k) Suppose that G is a graph. For vertices a and b define $a \sim b$ if and only if there is a path from a to b. Prove (with all the details) that $\sim$ is an equivalence relation on the set of vertices.

(l) If G is a connected graph and e is an edge then the graph obtained by removing e from G has at most two connected components.

(m) Euler's theorem for planar graphs: If G is a finite, planar, non-empty, connected graph with $V(G)$ vertices, $E(G)$ edges, and $F(G)$ faces, then $V(G) - E(G) + F(G) = 2$.

(n) Every fraction can be written in lowest terms.

(o) If A contains an injective sequence then there is a proper subset $B \subset A$ and a bijection $f\colon\ A \to B$.

(p) If there is a surjection $f\colon\ X \to Y$ then there is an injection $g\colon\ Y \to X$.

(q) If X is an infinite set then there is an injective sequence in X.

(r) $\operatorname{card} \mathbb{N} \times \mathbb{N} = \operatorname{card} \mathbb{N}$ (the Cantor Snake)

(s) The rationals are countable

(t) If X is countable and if $A \subset X$, then A is countable.

(u) The interval $(0,1)$ and the set $\mathbb{R}$ are uncountable

(v) For every set X, $\operatorname{card} X < \operatorname{card} \mathscr{P}(X)$.

(w) Let $\mathscr{F}$ be the set of functions from $X \to \{0,1\}$. Then $\operatorname{card}(\mathscr{P}(X)) = \operatorname{card} \mathscr{F}$.

(x) If Λ is a non-empty countable set and if for each $\lambda \in \Lambda$, A_λ is a countable set, then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is countable.

(8) Here are a few problems you haven't done before:

(a) Suppose that A and B are both countable sets. Then $A \cup B$ is countable.
(b) If $\operatorname{card} X = \operatorname{card} Y$ and $\operatorname{card} Y = \operatorname{card} Z$ then $\operatorname{card} X = \operatorname{card} Z$. (remember the precise definitions!)
(c) If $\operatorname{card} X = \operatorname{card} Y$ then $\operatorname{card} Y = \operatorname{card} X$.
(d) Let $A = \{a\}$ be a set with a single element and let X be any set. Let $\mathscr{F}$ be the set of functions from A to X. Prove that $\operatorname{card} \mathscr{F} = \operatorname{card} X$.

(e) Suppose that $A = \{a_1, a_2\}$ is a set with exactly two elements and let X be any set. Let $\mathscr{F}$ be the set of functions from A to X. Prove that $\operatorname{card} \mathscr{F} = \operatorname{card} X \times X$.