S16 MA 274: Constructing sequences and subsequences

In this worksheet, you (and your group) will construct prove one of the most important theorems in all of mathematics. Unfortunately, why it is so important is beyond the scope of this course, but hopefully you'll take an analysis, topology, or geometry class to find out!

We begin with some definitions:

Suppose that X is a set and that (x_n) is a sequence in X. A **subsequence** of (x_n) is a sequence (x_{n_k}) where (n_k) is a *strictly increasing* sequence in $\mathbb{N}$ (that is, $n_k < n_{k+1}$ for all k).

If $X = \mathbb{R}$, the sequence (x_n) is **increasing** if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ and **decreasing** if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. If the inequalities are strict, then we say (x_n) is **strictly increasing** or **strictly decreasing** respectively. If (x_n) is increasing or decreasing, we say it is **monotonic**.

If $A \subset \mathbb{R}$, then an **upper bound** for A is any $x \in \mathbb{R} \cup \{-\infty, \infty\}$ such that $a \leq x$ for all $a \in A$. The **least upper bound** or **supremum** of A is an upper bound $\sup A$ such that whenever x is an upper bound for A, $\sup A \leq x$. A **lower bound** for A is any $x \in \mathbb{R} \cup \{-\infty, \infty\}$ such that $x \leq a$ for all $a \in A$. The **greatest lower bound** or **infimum** of A is a lower bound $\inf A$ such that whenever x is a lower bound for A, $x \leq \inf A$. It is a theorem (which you may use) that every set $A \subset \mathbb{R}$ has an infimum and a supremum (which may be $\pm\infty$.)

A sequence (x_n) in $\mathbb{R}$ **converges** to $L \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$L - \varepsilon < x_n < L + \varepsilon.$$

Prove the following, by filling in the missing steps.

Theorem 0.1. Suppose that $A \subset \mathbb{R}$ and that $\sup A \neq \pm\infty$ (i.e. $\sup A \in \mathbb{R}$). Then A contains an increasing sequence converging to $\sup A$. If $\sup A \notin A$, then there is such a sequence which is strictly increasing.

Proof. We begin by defining the sequence and then we show it converges to $\sup A$. Since $\sup A \neq \pm\infty$, the set A is non-empty. Since $\sup A$ is the least upper bound for A, the number $\sup A - 1$ is not the least upper bound for A. That is, there exists $a_1 \in A$ such that

$$\sup A - 1 < a_1 \leq \sup A.$$

Assume that we have defined

$$a_1, a_2, \ldots, a_k$$

so that for each $j \in \{1, \ldots, k\}$

$$\sup A - \frac{1}{j} < a_j \leq \sup A.$$

$$\langle \text{Explain how to define } a_{k+1} \rangle.$$

Thus, by induction we have a sequence (a_n) so that for all $n \in \mathbb{N}$,

$$\sup A - \frac{1}{n} < a_n \leq \sup A.$$

$$\langle \text{Explain why } (a_n) \text{ is increasing. } \rangle.$$

$\langle$Explain how to change the proof to guarantee that if $\sup A \notin A$, then we can guarantee that (a_n) is strictly increasing. $\rangle$.

Now we show that (a_n) converges to $\sup A$. Let $\varepsilon > 0$. We must show that there is $N \in \mathbb{N}$ so that for all $n \geq N$,

$$\sup A - \varepsilon < a_n < \sup A + \varepsilon.$$

By basic properties of real numbers, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\frac{1}{n} < \varepsilon$. Then for all such n,

$$\sup A - \varepsilon < \sup A - \frac{1}{n} < a_n < \sup A < \sup A + \varepsilon.$$

Thus, (a_n) converges to A. □

The next three theorems can be proved by a slight modification of the previous proof, although we give a shorter proof of the first theorem.

Theorem 0.2. Suppose that $A \subset \mathbb{R}$ and that $\inf A \neq \pm\infty$. Then A contains a decreasing sequence converging to $\inf A$. If $\inf A \notin A$, then there is such a sequence which is strictly decreasing.

Proof. Let $-A = \{-a \in \mathbb{R} : a \in A\}$. Observe that $\inf A = -\sup(-A)$. Thus, by the previous theorem, there is a increasing sequence (x_n) in $-A$ converging to $\sup(-A)$. If $\sup(-A) \notin (-A)$, then the sequence may be taken to be strictly increasing. Since for all $n \in \mathbb{N}$, $x_n \in (-A)$, we have $-x_n \in A$. The sequence $(-x_n)$ is (strictly) decreasing if and only if (x_n) is (strictly) increasing. For all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, we have:

$$\sup(-A) - \varepsilon < x_n < \sup(-A) + \varepsilon.$$

Multiplying by (-1) we obtain:

$$\inf A + \varepsilon > -x_n > \inf A - \varepsilon.$$

Thus $(-x_n)$ is a (strictly) decreasing sequence in A converging to $\inf A$. □

Theorem 0.3. Suppose that (x_n) is a bounded sequence in $\mathbb{R}$. If (x_n) is increasing then it converges to $\sup \operatorname{range}(x_n)$. If (x_n) is decreasing then it converges to $\inf \operatorname{range}(x_n)$.

Proof. Suppose that (x_n) is increasing. Let $A = \{x_1, x_2, \ldots\}$ be the range of the sequence (x_n). We will show that (x_n) converges to $\sup A$. Let $\varepsilon > 0$ be given.

$\langle$Explain why there is an $N \in \mathbb{N}$ s.t. $\sup A - \varepsilon < x_N \leq \sup A\rangle$.

$\langle$Explain why for all $n \geq N, \sup A - \varepsilon < x_n \leq \sup A\rangle$.

The proof when (x_n) is decreasing is similar. □

The next proof actually uses subsequences.

Theorem 0.4. Suppose that (x_n) is a sequence in $\mathbb{R}$. Then (x_n) has a monotonic subsequence.

Proof. If (x_n) has a strictly increasing subsequence, then we are done. Assume, therefore, that (x_n) has no strictly increasing subsequence. Let $A_1 = \operatorname{range}(x_n)$.

$\langle$Explain why $\sup A_1 \in A_1$.

Since $\sup A_1 \in A_1$, there exists $n_1 \in \mathbb{N}$, such that $x_{n_1} = \sup A_1$. Let $A_2 = \{x_n : n > n_1\}$.

$\langle$Explain why $\sup A_2 \in A_2$ and $\sup A_2 \leq x_{n_1}\rangle$

Assume that we have defined $n_1 < n_2 < \cdots < n_k$ and $A_1, A_2, \ldots, A_k$ so that for all $j \in \{1, \ldots, k\}$:

(i) $A_j = \{x_n : n > n_{j-1}\}$
(ii) $x_{n_j} = \sup A_j$

$$\langle \text{Define } A_{k+1} \text{ and } n_{k+1} \rangle$$

By induction we have a sequence (n_k) of $\mathbb{N}$.

$$\langle \text{Explain why } (n_k) \text{ is increasing. } \rangle$$

Thus (x_{n_k}) is a subsequence of (x_n).

$$\langle \text{Explain why } (x_{n_k}) \text{ is decreasing. } \rangle$$

Thus, (x_n) has a monotonic subsequence. □