

Practice Exam 1

You may not use textbooks, notes, or refer to other people (except the instructor). Please do not place any answers on the exam itself. Instead turn in your final answers in the proper order, and with all preliminary work clearly labelled and attached at the very end of what you turn in. Staple this cover sheet as the first page of your exam.

Problem	Score	Possible
1		20
2		15
3		10
4		5
5		10
6		10
7		10
8		10
9		10
10		10
11		10 (extra-credit)
Total		120

1. DEFINITIONS

A number $p \in \mathbb{N}$ is **prime** if $p \neq 1$ and if p is a multiple of only itself and 1.

A number q is **rational** if it can be written as the fraction of integers.

A **group** is a set G with an operation \circ such that the following axioms hold:

(G1) For all $a, b \in G$, $a \circ b \in G$.

(G2) There exists $\mathbf{1} \in G$ (called the “identity”) such that for all $a \in G$,

$$a \circ \mathbf{1} = \mathbf{1} \circ a = a.$$

(G3) For all $a \in G$, there exists an element a^{-1} (called the “inverse” of a) such that

$$a \circ a^{-1} = a^{-1} \circ a = \mathbf{1}.$$

(G4) For all $a, b, c \in G$,

$$(a \circ b) \circ c = a \circ (b \circ c).$$

A **subgroup** of a group is a subset that is also a group (using the same operation).

- (1) (15 pts) Give a precise definition of the following terms:
- negation
 - free variable
 - complement of a set
- (2) (20 pts) Write the negations of the following statements. Phrase your answer as positively as possible.
- There exists a natural number n such that \sqrt{n} cannot be written as the ratio of two natural numbers.
 - For every $x \in \mathbb{N}$, $0 + x = 0$.
 - Let (s_n) be a sequence and L a real number. For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|s_n - L| < \varepsilon$.
 - If $n \in \mathbb{N}$ has the property that n^3 is a multiple of 3, then n is a multiple of 7.
- (3) (10 pts) Let Λ be an index set, and for each $\lambda \in \Lambda$, let A_λ be a set. Prove that
- $$\left(\bigcap_{\lambda \in \Lambda} A_\lambda \right)^c \subset \bigcup_{\lambda \in \Lambda} A_\lambda^c.$$
- (This is one part of DeMorgan's Laws. You cannot, of course, refer to those in your proof.)
- (4) (5 pts) Let $X = \{a, b\}$. List all elements of $\mathcal{P}(X)$ (the power set of X) **and also** 5 elements of $\mathcal{P}(\mathcal{P}(X))$.
- (5) (10 pts) Prove that the graphs of the equations $y = x^2 + 7$ and $y = 2x + 4$ in \mathbb{R}^2 do not intersect.
- (6) (10 pts) Suppose that G is a group with operation \circ . Prove that the identity in G is unique.
- (7) (10 pts) Prove that if n is a natural number, then n is even or odd.

(8) (10 pts) Let $A \subset \mathbb{N}$. Prove: If $A \neq \emptyset$, then A has a least element.

(This is the well-ordering principle. You may not refer to that result in your proof.)

(9) (10 pts) Let p be a positive prime number. Prove that \sqrt{p} is irrational.

(In your proof you may assume basic arithmetic facts. Any other fact that you assume without proof should be separated out as a separate lemma.)

(10) (10 pts) Let G be a group with operation \circ . Suppose that $\mathcal{U} \subset \mathcal{P}(G)$ is a set all of whose elements are subgroups of G . Prove that $\bigcap \mathcal{U}$ is a subgroup of G .

(11) (10 pts Extra-Credit) Let $a, b \in \mathbb{N}$. Prove that there exist unique $q, r \in \mathbb{Z}$ with $0 \leq r < b$ such that $a = bq + r$.

(This is called the division algorithm. Hint: Consider the set

$$S = \{a - bk : k \in \mathbb{N} \text{ and } a - bk \geq 0\}$$

Apply the Well-Ordering Principle.)