S16 MA 274: Group Project 3

You will work in teams of 2 or 3 to complete this project. Each person in the group must turn in (on Monday **April 4**) the following **typed** in LaTeX. You may hand-draw any pictures that you need. The proofs must be found through group discussion, but the write-ups should be done independently. You should not consult any sources other than your course notes or texts.

(1) Statement and proofs of the statements below. See group project 2 (on the reverse side) for definitions.
(2) A paragraph naming each person in the team, and summarizing the process by which the team came to find the proofs. You should address such questions as: How difficult was it to figure out what to prove? Did the team talk through everything together or did they work independently and then compare solutions? Did one person (who?) take the lead or provide all the ideas? Did you feel comfortable questioning your teammates solutions? why or why not? What did you contribute to the process? You should also compare your work on this project to the work on group project 2 (on which this one was based).

The Project:

(1) (This was a HW problem) Suppose that G is a connected graph and that v and w are distinct vertices of G. Prove that there is a path from v to w with no backtracking.

(2) Prove that if G is a connected graph and if e is an edge of G that does not disconnect G, then G contains a cycle $v_0, \dots, v_n$ with $n \geq 1$.

<u>Hint</u>: Let v_- and v_+ be the endpoints of e. Choose a path in $G-e$ from v_- to v_+ without backtracking (using the previous part). Put in the edge e to create a cycle in G. (Be sure to be precise about the details.)

(3) Explain why the previous part shows that if G is a connected graph without without cycles of positive length, then every edge of G separates G.

---

(4) Suppose that e is an edge of a connected graph. Let $G-e$ be the result of removing e from G. Let v_- and v_+ be the endpoints of e. Prove that for any vertex w of $G-e$ there is a path in $G-e$ from w to either v_- or v_+.

(5) Use the previous part to prove that if e is an edge of a connected graph G, then $G-e$ is connected if and only if there is a path in $G-e$ between the endpoints of e.

(6) Let G be a graph with a cycle $v_0, \dots, v_n$ with $n \geq 1$. Prove that G contains a cycle $w_0, \dots, w_k$ with $k \geq 1$ such that if $w_i = w_j$ then $\{i, j\} \in \{0, k\}$. (No vertex repetitions)

<u>Hint</u>: Use the Well-Ordering principle: Suppose that $v_0, \dots, v_n$ is some cycle in G. Let

$$S = \{m \in \mathbb{N} : \exists i \in \{0, \dots, n-1\} \text{s. t. } v_i = v_{i+m}\}$$

Explain why $S \neq \varnothing$. Choose k to be the least element of S and let $i \in \{0, \dots, n-1\}$ be the corresponding index such that $v_i = v_{i+k}$. Define $w_j = v_{j+i}$ and explain why $w_0, w_1, \dots, w_k$ is a cycle without vertex repetitions.

(7) Suppose that G is a connected graph containing a cycle of positive length. Prove that G has an edge e such that $G-e$ is connected.

Hint: Use the previous result to conclude that G has a cycle of positive length without vertex repetitions. Let e be any edge of that cycle. Prove $G-e$ is connected by showing that there is a path in $G-e$ between the endpoints of e.

(8) Explain why the previous part shows that if G is a connected graph such that every edge of G separates G then G has no cycle of positive length.

Group Project 2 (for reference)

The word "graph" in mathematics has several meanings. One of the meanings is that a graph G consists of a set of vertices (or nodes) and a set of edges (or connections). We'll call the set of vertices $V(G)$ and the set of edges $E(G)$. If $v,w \in V(G)$ are distinct then the set $\{v,w\}$ is called the **edge** between v and w. Every element of $E(G)$ is a set of the form $\{v,w\}$ where v and w are distinct elements of $V(G)$.

If $e = \{v,w\} \in E(G)$, we can form a new graph denoted $G-e$ by letting $V(G-e) = V(G)$ and $E(G-e) = \{f \in E(G) : f \neq e\}$. The graph $G-e$ is said to be obtained by **removing** the edge e from $E(G)$.

A **path** in a graph G is a list of vertices

$$v_0, v_1, v_2, \ldots, v_n$$

(for some $n \in \mathbb{N} \cup \{0\}$) such that for each $i \in \{0,\ldots,n-1\}$, the set $\{v_i, v_{i+1}\} \in E(G)$. (That is, adjacent vertices in the path are endpoints of an edge in G.) We say that the path is **from** v_0 to v_n. The path is a **cycle** if $v_0 = v_n$ and if for all $i \in \{1,\ldots,n-1\}$, $v_{i-1} \neq v_{i+1}$ (i.e. no "back-tracking").

A graph is **connected** if whenever $v,w \in V(G)$, there exists a path from v to w. If G is connected and if $e \in E(G)$ but $G-e$ is disconnected, then we say that e **disconnects** G.

Theorem. Suppose that G is a connected graph. Then every edge of G disconnects G if and only if G does not have a cycle.

Group Assignments for Group Project 3

- Mirco Dinelli, Abby Johnson, Luc Marrie
- Sarah Whitey, Katie Discipio
- Zhuofan Zhang, Stacey Hou, Makoto Kinoshita
- Katherine Zafirson, Eric Hartman, Hanjing Shi
- Hannah Bossi, Pearson Treanor, John Baron
- Keith Barnatchez, Joshua Young, Zitong Wang
- Jason Beland, Grace Farnkoff, Tomotaka Cho
- Robert Cambell, Xuning Wang
- Chowdhury Farabee, Benard Kibet, Avery Heilbron