S15 MA 274: Exam 3 Study Questions

You can find solutions to some of these problems on the next page. These questions only pertain to material covered since Exam 2. The final exam is cumulative, so you should also study earlier material.

(1) Know the precise definitions of the terms requested for your journal.

(2) Be able to prove that a sequence (s_i) in a set X either has a constant subsequence or has a subsequence of distinct terms.

(3) Know how to prove that if a set contains a surjective sequence which has a subsequence of distinct terms, then the sequence has a subsequence which is both surjective and of distinct terms. Explain why this shows that if there is a surjection $\mathbb{N} \twoheadrightarrow X$ from $\mathbb{N}$ onto an infinite set, then $\mathrm{card}(\mathbb{N}) = \mathrm{card}(X)$.

(4) Be able to prove that an infinite set contains a sequence of distinct terms.

(5) Be able to prove that there is a bijection from an infinite set to a proper subset.

(6) Be able to prove that $\mathbb{N} \times \mathbb{N}$ and $\mathbb{Q}$ are countable.

(7) Be able to prove that the countable union of countable sets is countable.

(8) Be able to prove that $\mathbb{R}$ is uncountable and (even more!) that it has the same cardinality as $\mathcal{P}(X)$.

(9) Be able to prove that the set of irrational numbers is uncountable.

(10) Be able to prove that if X is a set then $\mathrm{card}(X) < \mathrm{card}(\mathcal{P}(X))$.

(11) Be able to prove that the set of binary sequences has the same cardinality as $\mathcal{P}(\mathbb{N})$.

(12) Know how to prove that if X is a set, then the set of characteristic functions (functions $X \to \{0, 1\}$) on X has the same cardinality as $\mathcal{P}(X)$.

(13) Be able to prove (using the axiom of choice) that if there is a surjection $Y \to X$ then $\mathrm{card}(X) \leq \mathrm{card}(Y)$

(14) Know how to prove that if X is a set with at least two distinct elements then the set of all sequences in X is uncountable.

(15) Be able to prove that the set of algebraic numbers is countable.

(16) (A new one!) Suppose that X is a countable set. Let $\mathcal{C}$ be the set of sequences in X which are eventually constant. That is, there exists $a \in X$ such that for every $(x_n) \in \mathcal{C}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n = a$. Prove that $\mathcal{C}$ is countable.

A couple solutions:

Theorem. Suppose that (s_i) is a surjective sequence in a set X and that (s_i) has a subsequence of distinct terms. Then (s_i) has a subsequence of distinct terms which is surjective.

Proof. We define the desired subsequence (s_{i_n}) recursively. Let $i_1 = 1$, so that $s_{i_1} = s_1$. Assume now that we have defined $i_1, \ldots, i_n$ so that the following hold:

(P(n) $i_1 < i_2 < \cdots < i_n$
(Q(n) $s_{i_1}, s_{i_2}, \cdots, s_{i_n}$ are all distinct
(R(n)) If $j \leq i_n$, then there exists $k \leq n$ such that $s_j = s_{i_k}$

Condition P(n) guarantees that we are building a subsequence of (s_i). Condition Q(n) will guarantee that we have a subsequence of distinct terms. Condition R(n) will guarantee that our subsequence is surjective, as won't have skipped any terms of (s_i), which is a surjective sequence.

We now show how to define i_{n+1} so that properties P($n+1$), Q($n+1$), and R($n+1$) still hold. Let

$$S_n = \min(j : j > i_n \text{ and } s_j \text{ is distinct from each of } s_{i_1}, \ldots, s_{i_n}).$$

The set S_n is non-empty since (s_i) contains a subsequence of distinct terms. Thus, by the well-ordering principle, $i_{n+1} = \min S_n$ exists. Since $i_{n+1} \geq i_n$, property P($n+1$) holds. Since $s_{i_{n+1}}$ is distinct from each of $s_{i_1}, \ldots, s_{i_n}$, property Q($n+1$) holds. We now show that R($n+1$) holds.

Suppose that $j \leq i_{n+1}$. If $j \leq i_n$, then by R(n), there exists $k \leq n < n+1$ so that $s_j = s_{i_k}$, as desired. Suppose, therefore, that $i_n < j \leq i_{n+1}$. If $j \notin S_n$, then by the definition of S_n, there exists one of $k \leq n < n+1$ such that $s_j = s_{i_k}$, as desired. If $j \in S_n$, then $i_{n+1} \leq j$, since i_{n+1} is the minimal element of S_n. Thus, $i_{n+1} = j$ and so $s_{i_{n+1}} = s_j$, as desired. Consequently, R($n+1$) holds.

By induction we have a sequence (s_{i_n}) such that for each n, P(n), Q(n), and R(n) hold. In particular,

$$i_1 < i_2 < \cdots$$

which means that (s_{i_n}) is, in fact, a subsequence of (s_i). If $s_{i_n} = s_{i_m}$ then either $n \leq m$ or $m \leq n$. If $n \leq m$, then by Q(m), we must have $i_n = i_m$. Likewise, if $m \leq n$, then by Q(n), we must have $i_m = i_n$. Consequently (s_{i_n}) is a subsequence of distinct terms.

Suppose that $x \in X$. Since the original sequence is surjective, there exists $j \in \mathbb{N}$ such that $s_j = x$. Choose n large enough so that $j \leq i_n$. Then by

R(n), there exists k such that $s_{i_k} = s_j = x$. Hence, (s_{i_n}) is a surjective subsequence of distinct terms in X. □

Since sequences are functions with domain $\mathbb{N}$, this theorem implies that if there is a surjective function $\mathbb{N} \twoheadrightarrow X$ and if X is infinite then (by Theorem 7.2.5) there is a bijection (since subsequences of distinct terms are injective functions and surjective subsequences are surjective functions) from $\mathbb{N}$ to X.

Theorem. Suppose that Λ is a non-empty countable set and that for all $\alpha \in \Lambda$, the set A_α is countable and non-empty. We prove that $\bigcup_{\alpha\in\Lambda} A_\alpha$ is countable.

Proof. Since Λ is countable, there exists a surjection $f\colon \mathbb{N} \to \Lambda$. Since, for each $\alpha \in \Lambda$, the set A_α is countable, there exists a surjection $g_\alpha\colon \mathbb{N} \to A_\alpha$.

Define $h\colon \mathbb{N} \times \mathbb{N} \to \bigcup_{\alpha\in\Lambda} A_\alpha$ by $h(i,j) = g_{f(i)}(j)$. We claim that h is a surjection. Suppose that $a \in \bigcup_{\alpha\in\Lambda} A_\alpha$. Then there exists $\alpha \in \Lambda$ such that $a \in A_\alpha$. Since f is surjective, there exists $i \in \mathbb{N}$ such that $f(i) = \alpha$. Since $g_{f(i)} = g_\alpha$ is surjective, there exists $j \in \mathbb{N}$ such that $g_{f(i)}(j) = a$. Consequently, $h(i,j) = a$ and so h is surjective.

Since $\mathbb{N} \times \mathbb{N}$ is countable, there exists a surjection $k\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Then the function $h \circ k\colon \mathbb{N} \to \bigcup_{\alpha\in\Lambda} A_\alpha$ is a surjection since the composition of surjections is surjective. Since there is a surjection from $\mathbb{N}$ to $\bigcup_{\alpha\in\Lambda} A_\alpha$, the set $\bigcup_{\alpha\in\Lambda} A_\alpha$ is countable. □

Theorem. If there is a surjection $Y \to X$ then $\operatorname{card}(X) \le \operatorname{card}(Y)$.

Proof. Suppose that $f\colon Y \to X$ is a surjection. We will produce an injection $g\colon X \to Y$. The existence of such an injection guarantees that $\operatorname{card}(X) \le \operatorname{card}(Y)$.

For each $x \in X$, let $U_x = f^{-1}(x)$. Since f is a surjection, each U_x is nonempty. By the axiom of choice, we can choose a unique element y_x in each set U_x. Define $g(x) = y_x$. Since y_x was chosen uniquely, the relation g is a function. Suppose that $g(x_1) = g(x_2)$. Then $y_{x_1} = y_{x_2}$. Hence $y_{x_2} \in U_{x_1}$ and $y_{x_1} \in U_{x_2}$. But we chose a unique y_x in each U_x, so $U_{x_1} = U_{x_2}$. Since $U_{x_1} = f^{-1}(x_1)$ and $U_{x_2} = f^{-1}(x_2)$, and since f is a function we must have $x_1 = x_2$. □

Theorem. Suppose that X is a set with at least two distinct elements. Then the set of all sequences in X is uncountable.

Proof. Let $\mathcal{S}$ be the set of all sequences in X. We will show that there exists a surjection f from $\mathcal{S}$ to the set of binary sequences. Since the set of binary sequences has the same cardinality as $\mathcal{P}(\mathbb{N})$, by the previous theorem, we will have $\mathrm{card}(\mathcal{P}(X)) \leq \mathrm{card}(\mathcal{S})$. Given that, suppose that $\mathcal{S}$ is countable. Then

$$\mathrm{card}(\mathcal{P}(\mathbb{N})) \leq \mathrm{card}(\mathcal{S}) \leq \mathrm{card}(\mathbb{N}).$$

In which case $\mathrm{card}(\mathcal{P}(\mathbb{N})) \leq \mathrm{card}(\mathbb{N})$, a contradiction. Thus, we will be done if we can prove the existence of f.

Let $x_1 \in X$. Define $g\colon\ X \to \{0,1\}$ by

$$g(x) = \begin{cases} 1 & x = x_1 \\ 0 & x \neq x_1. \end{cases}$$

Let B be the set of binary sequences. Recall that if $s \in \mathcal{S}$, then s is a function with domain $\mathbb{N}$ and codomain X. Define $f\colon\ \mathcal{S} \to B$ by $f(s) = g \circ s$. That is, $f(s)$ is the sequence that takes the value 1 whenever s hits x_1 and takes the value 0 whenever s doesn't hit x_1.

We now show that f is a surjection. Let $b\colon\ \mathbb{N} \to \{0,1\}$ be a binary sequence. Since X has at least two elements, there exists $x_0 \in X\backslash\{x_1\}$. Define a sequence $s_b\colon\ \mathbb{N} \to X$ as follows

$$s_b(n) = \begin{cases} x_0 & b(n) = 0 \\ x_1 & b(n) = 1 \end{cases}$$

Notice that $f(s_b) = b$, so f is surjective. $\square$