MA 274: Exam 2 Study Guide

(1) Know the precise definitions of the terms requested for your journal.

(2) Review proofs by induction.

(3) Prove that it is impossible to write a computer program that can determine if other computer programs contain an infinite loop. (i.e. the halting problem)

(4) Prove that $A \times B = \{(a, b) : a \in A, b \in B\}$ is a set (using the axioms) if A is a set and if B is a set.

(5) Be able to use the definition of $+$ on the natural numbers to prove basic facts about $+$. (You will, however, not be asked to prove that $+$ is commutative or associative.)

(6) Be able to prove that something is or isn't an equivalence relation.

(7) Be able to prove that something is or isn't a partial order.

(8) Understand what it means to prove that a function on equivalence classes is well-defined.

(9) Be able to prove all or portions of the following facts. You should also study other homework problems

    (a) The intersection of inductive sets is inductive.

    (b) There exists a unique smallest inductive set.

    (c) There does not exist a set of all sets.

    (d) 2 + 2 = 4

    (e) If $a, b, c \in \mathbb{N}_0$, then $b + a = c + a$ implies $b = c$.

    (f) If X is a set, then the relation

    $$(A \leq B) \Leftrightarrow (A \subset B)$$

    is a partial order on $\mathcal{P}(X)$.

    (g) Prove that equivalence classes form a partition.

    (h) Prove that if $\sim$ is an equivalence relation, then $x \sim y$ if and only if $[x] = [y]$.

(i) If G is a group and if H is a subgroup, then $\sim$ is an equivalence relation on G where

$$(x \sim y) \Leftrightarrow y^{-1} \circ x \in H$$

(j) Using the previous equivalence relation, for all $x \in G$, prove that there exists a bijection from $[x]$ to H.

(k) State and prove LaGrange's Theorem.

(l) The compositions of injective (or surjective or bijective) functions is injective (or surjective or bijective).

(m) A function $f\colon X \to Y$ is a bijection if and only if it has an inverse function $f^{-1}\colon Y \to X$. (That is there is a function $f^{-1}\colon Y \to X$ such that $f^{-1} \circ f(x) = x$ for all $x \in X$ and $f \circ f^{-1}(y) = y$ for all $y \in Y$.)

(10) Here are some new facts for you to try to prove:

(a) Suppose that $f\colon X \to Y$ is a function. If $A \subset X$, we define $f(A) = \{y \in Y : \exists a \in A f(a) = y\}$. Suppose that A, B are subsets of X. Prove that $f(A \cup B) = f(A) \cup f(B)$. Give an example to show that $f(A \cap B)$ need not be equal to $f(A) \cap f(B)$.

(b) Let $X = \mathcal{P}(\mathbb{R})$ and define $\sim$ on X by $A \sim B$ if and only if there exists a bijection $f\colon A \to B$. Prove that $\sim$ is an equivalence relation.

(c) Let $(X, \leq)$ and $(Y, \prec)$ be sets with partial orders. Define a partial order $\ll$ on $X \times Y$ by:

$(a, b) \ll (c, d) \Leftrightarrow (a \leq c)$ and if $a = c$ then $b \prec d$

Prove that $\ll$ is a partial order and explain how it is related to finding words in a dictionary.

(d) Prove using induction that the number of permutations of a set of n elements is $n!$. (A permutation is a bijection from a set to itself.)

(e) Suppose that $f\colon X \to X$ is a bijection on a set with n elements. Prove that there exist transpositions $f_1, \ldots, f_k$ of X such that $f = f_k \circ f_{k-1} \circ \ldots f_2 \circ f_1$. (A transposition is a bijection that simply swaps two elements and leaves all other elements unchanged.) Hint: Induct on n.

(f) Give an example of a permutation of $\mathbb{N}$ which is not the composition of a finite number of transpositions.