MA 274: Circle Rotations

Let $S^1 = \{x \in \mathbb{R}^2 : ||x|| = 1\}$ be the unit circle in $\mathbb{R}^2$ and let $T_\theta : S^1 \to S^1$ be a rotation of the circle by θ radians counter-clockwise. Let $x_0 = (1,0) \in S^1$. Observer that if $k \in \mathbb{Z}$ then $T^k : S^1 \to S^1$ rotates the circle by an angle of $k\theta$. Observe that two angles θ and θ' differ by an integer multiple of 2π, if and only if $T_\theta = T_{\theta'}$.

Lemma 1. If θ is not a rational multiple of 2π, then for all $k \in \mathbb{Z} \setminus \{0\}$, $T_\theta^k(x_0) \neq x_0$.

Proof. Suppose, to the contrary, that there exists $k \in \mathbb{Z} \setminus \{0\}$, such that $T_\theta^k(x_0) = x_0$. Since a rotation that doesn't move one point doesn't move any point, the function $T_\theta^k : S^1 \to S^1$ is the identity function. In other words, there is $m \in \mathbb{Z}$ such that $k\theta = 2\pi m$. Since $k \neq 0$ we have $\theta = \frac{m}{k}(2\pi)$, contrary to our assumption on θ. Thus, for all $k \in \mathbb{Z} \setminus \{0\}$, $T_\theta^k(x_0) \neq x_0$. □

Theorem 2. Suppose that θ is not a rational multiple of 2π. Then for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $T_\theta^k(x_0)$ is within a distance of ε from x_0 on S^1.

Proof. We prove the following claim by induction on n:

Claim: For every $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that the distance from $T_\theta^k(x_0)$ to x_0 on S^1 is at most $|\theta|/2^n$.

Base Case: $n = 1$.

For each $k \in \mathbb{N}$, consider the open interval I_k on S^1 between $T_\theta^k(x_0)$ and $T_\theta^{k+1}(x_0)$. This interval has length θ since T rotates S^1 by an angle of θ and S^1 is the unit circle. By the Lemma since x_0 is never an endpoint of I_k for any k, there is a minimal m such that $x_0 \in I_k$. We note that x_0 is not the midpoint of the interval, for then $\theta/2$ would be an integer multiple of 2π which contradicts the fact that θ is not a rational multiple of 2π. Thus, either $T_\theta^m(x_0)$ or $T_\theta^{m+1}(x_0)$ is within $\theta/2$ of x_0. Thus for $k = m$ or $k = m+1$ we have our Base Case.

Inductive Step: Assume that for some $n \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that $T_\theta^k(x_0)$ is within $\theta/2^n$ of x_0. We will prove that there is a $p \in \mathbb{N}$ such that $T_\theta^p(x_0)$ is within $\theta/2^{n+1}$ of x_0.

Let ψ be the angle (lying between $(-\pi, \pi)$ from x_0 to $T_\theta^k(x_0)$ so that $|\psi|$ being the distance from x_0 to $T_\theta^k(x_0)$ along S^1 is strictly less than $\theta/2^n$. Observe that $T_\theta^k = T_{k\theta} = T_\psi$ and that the number ψ is, therefore, not a rational multiple of 2π. Applying the Base Case to ψ in place of θ, we see that there exists $a \in \mathbb{N}$ such that the distance from $T_\psi^a(x_0)$ to x_0 along S^1 is strictly less than $\psi/2 = \theta/2^{n+1}$. Since

$$T_\psi^a(x_0) = (T_\theta^k(x_0))^a = T_\theta^{ak}(x_0)$$

is within $\theta/2^{n+1}$ of x_0, letting $p = ak$ we have our result. This completes the proof of the Claim.

Since $\varepsilon > 0$, there is an $n \in \mathbb{N}$ such that $|\theta|/2^n < \varepsilon$. By the Claim, there exists $k \in \mathbb{N}$ such that $T_\theta^k(x_0)$ is within $|\theta|/2^n$ (and thus within ε) of x_0, as desired. □

We now show that we can approximate *any* point using the images of x_0 under iterations of T_θ. The points of the sequence $(T_\theta^k(x_0))$ are called **iterates** of x_0 under T_θ.

Theorem 3. For every $x \in S^1$ and for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $T^k(x_0)$ is within ε of x.

Proof. By Theorem 2 there exists $m \in \mathbb{N}$ such that $T^m(x_0)$ is within ε of x_0. Let I be the closed interval on S^1 between $T^m(x_0)$ and x_0. Applying T_θ to I enough times, covers all of S^1 with copies of I. The endpoints of these copies of I are iterates under T_θ of x_0. The point x lies in (at least) one of these intervals and so there is an iterate of x_0 under T_θ which is within ε of x. □

Question: Is it possible that there is some θ so that *every* point of S^1 is an iterate of x_0 under T_θ? What about being an iterate under either T_θ or $T_\theta^{-1} = T_{-\theta}$?

Finally, some terminology. We can define a sequence (x_n) recursively by defining, for all $n \in \mathbb{N}$, $x_n = T_\theta(x_0)$ (recalling that $x_0 = (1,0)$.) The sequence is an example of an **iterated function sequence**. If $X \subset S^1$ we say that X is **dense** in S^1 if every open interval in S^1 contains a point of X. Theorem **??** shows that if θ is irrational then the points of the iterated function sequence (x_n) are dense in S^1.