Axioms and Geometry

Plimpton 322

c. 1700 BCE

Euclid of Alexandria
(325? BCE - 265? BCE)

Almost from the time of its writing and lasting almost to the present, the Elements has exerted a continuous and major influence on human affairs. It was the primary source of geometric reasoning, theorems, and methods at least until the advent of non-Euclidean geometry in the 19th century. It is sometimes said that, next to the Bible, the "Elements" may be the most translated, published, and studied of all the books produced in the Western world.

B.L. van der Warden

In the course of my law reading I constantly came upon the word "demonstrate". I thought at first that I understood its meaning, but soon became satisfied that I did not. I said to myself, What do I do when I demonstrate more than when I reason or prove? How does demonstration differ from any other proof?

I consulted Webster's Dictionary. They told of 'certain proof,' 'proof beyond the possibility of doubt'; but I could form no idea of what sort of proof that was. I thought a great many things were proved beyond the possibility of doubt, without recourse to any such extraordinary process of reasoning as I understood demonstration to be. I consulted all the dictionaries and books of reference I could find, but with no better results. You might as well have defined blue to a blind man.

At last I said,- Lincoln, you never can make a lawyer if you do not understand what demonstrate means; and I left my situation in Springfield, went home to my father's house, and stayed there till I could give any proposition in the six books of Euclid at sight. I then found out what demonstrate means, and went back to my law studies.

Euclidean Axioms

1. For every two distinct points, there exists a unique line that passes through them.

2. Given a finite straight line, it is possible to continue it, as a straight line.

3. Given a point and a radius, there exists a circle with that center and radius

4. All right angles equal one another.

5. --------

Euclidean Axioms

"One must be able to say at all times -- instead of points, straight lines, and planes -- tables, chairs and beer mugs." (Hilbert)

The Fifth Postulate

If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.
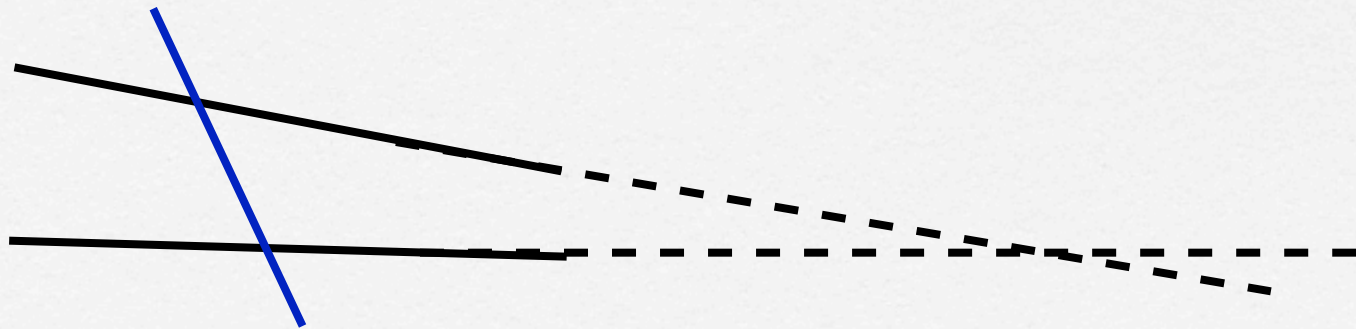
The Fifth Postulate (Playfair)

For every line L and every point P not on L, there is exactly on line through P parallel to L.
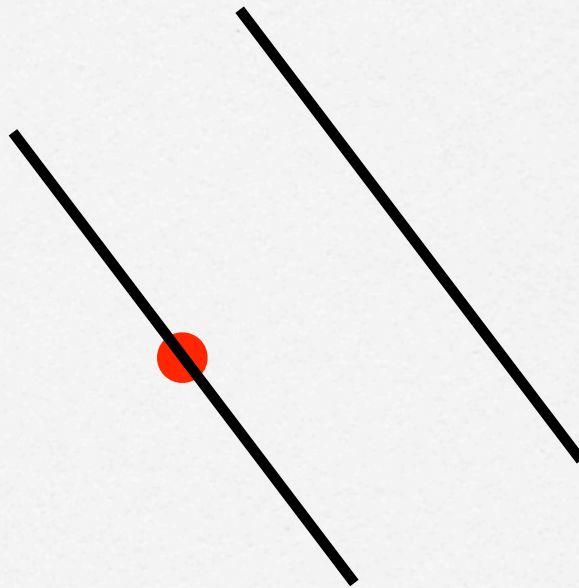
What if .... ?

There exists a line L and a point P not on the line such that there exist at least two distinct lines parallel to L and passing through P.

Nicolai Lobachevsky
1829

János Bolyai
1823

Axioms of Hyperbolic Geometry

1. Given two distinct points, there is a unique line containing them.

2. Straight lines can be continued indefinitely.

3. Given a point and a line segment having that point as an endpoint, there is a circle with that center and radius.

4. All right angles are equal.

5. There exists a line L and a point P not on L, such that there are at least two lines through P parallel to L.
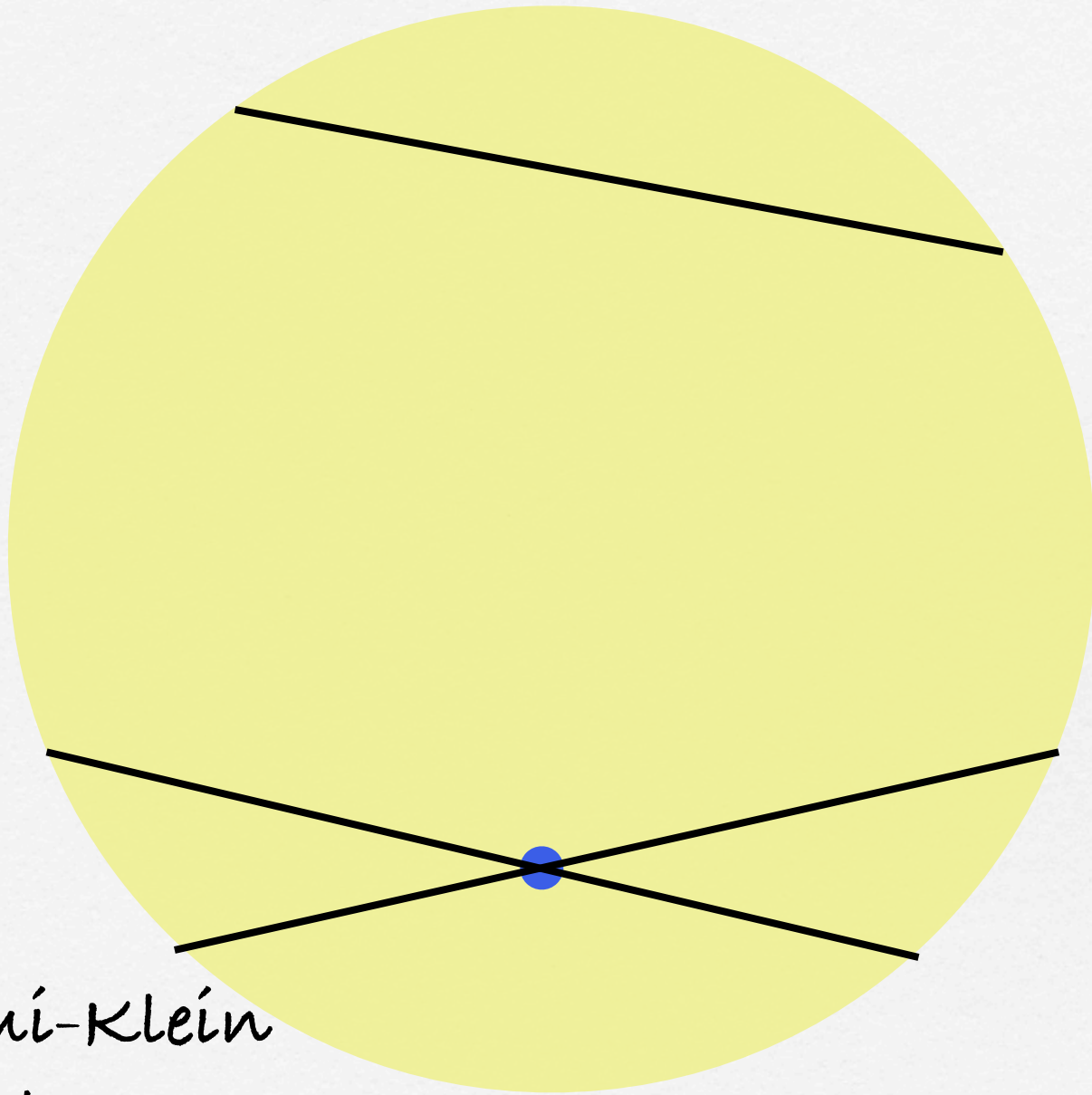
The Beltrami-Klein
Model

Felix Klein's Erlanger Programme

A *geometry* is a space and a <u>group</u> of symmetries (called isometries).

Hyperbolic geometry is the open unit disc and the group of 2x2 matrices with entries in the complex numbers having determinant one (modulo center).

Euclidean geometry is the plane and the group of orthogonal real 2x2 matrices together with translations.