Probability Study Guide

MA 111 Spring 2015

1. General Advice

The final exam is cumulative but will be the same length as the first two exams. You will have 3 hours to take it. Remember to study:

- Symmetry and Group Theory
- *Flatland* and the mathematics of higher dimensions
- Probability

Use the following tools for studying:

- Class notes (both from class and the ones posted online)
- Problem Sets
- Questions from reading assignments
- Discussion Questions from in Class
- Review the readings
- Study Guides from exams 1 and 2, in addition to this one.

What follows is a study guide for the material since the second exam.

2. General Concepts

(1) Know the definitions of: sample space, probability space, outcome, event, experiment, conditional probability, independent events, expected value, frequentist probability, subjectivist probability.
(2) Calculate the probability that somebody wins an unfinished game.
(3) Be able to summarize Pascals difficulties with Fermats solution to the unfinished game.
(4) Be able to summarize how Pascal set about trying to understand Fermats solution.
(5) Calculate the probability of an event when you know the probabilities of each outcome.
(6) If event B is independent of event A, be able to calculate the probability of the event A then B.
(7) Calculate the probability of an event given additional information.
(8) Calculate average wins or losses if a betting game is played many times.
(9) Be able to explain the calculations involved in the birthday paradox.

(10) Understand and be able to explain the prosecutors fallacy (see Devlin's book).
(11) Given two conditional probabilities, be able to figure out which is relevant to a particular situation. Given a situation, be able to figure out which conditional probability is relevant.
(12) Know who Pascal, Fermat, Graunt, Bayes, Cardano, Galileo, Huygens are and something about their contributions to probability and statistics (see Devlin's book)

3. More sample problems

Most of the questions concern the experiment where two fair dice are rolled and the results are added.

(1) What is the probability space for that experiment?

(2) Suppose that the experiment is repeated twice. What is the probability that a 4 is obtained both times? Why?

(3) Suppose the experiment is repeated 10 times. What is the probability that exactly 6 out of the 10 times result in a 7?

(4) Suppose that the experiment is repeated 10 times. What is the probability that at least 2 out of the 10 times result in an odd number?

(5) Suppose that the experiment is repeated 10 times and that at least 2 of the 10 times resulted in an odd number. What is the probability that exactly 6 out of the 10 times resulted in a 7?

(6) Suppose that Alf and Bettina are betting on the outcome of the experiment. If a number 2 - 6 is obtained, Bettina gives Alf $10. If a number 8 - 12 is obtained, Alf gives Bettina $10. If a 7 is rolled, they each give $2 to charity. If the game is repeated many times, how much on average will Bettina win or lose?

(7) Suppose that Alf and Bettina have played the game (from the previous problem) 4 times and that Bettina has won exactly 3 out of the 4. What is the probability that if they play the game 2 more times (for a total of 6 times) that she would win at least 4 out of the 6 games?

(8) Consider 2 events. Event E is the event that some smokes Marijuana. Event F is the event that someone uses Heroine.

(a) Suppose you want to figure out whether or not Marijuana is a "gateway drug"; that is, whether someone who uses Marijuana is likely to use Heroine. Should you calculate $P(E|F)$ or $P(F|E)$? Why?
(b) If you want to study the drug habits of Heroine users, should you calculate $P(E|F)$ or $P(F|E)$? Why?
(c) Explain why the difference between $P(E|F)$ and $P(F|E)$ is important.

(9) State the Monty Hall problem and explain why it is better to switch doors.

(10) Give a complete, precise statement of Pascal's Wager.

Solutions

(1) What is the probability space for that experiment?

Solutions: The sample space is $\{2,3,4,5,6,7,8,9,10,11,12\}$. The probabilities are:

P(2)	=	1/36
P(3)	=	2/36
P(4)	=	3/36
P(5)	=	4/36
P(6)	=	5/36
P(7)	=	6/36
P(8)	=	5/36
P(9)	=	4/36
P(10)	=	3/36
P(11)	=	2/36
P(12)	=	1/36

(2) Suppose that the experiment is repeated twice. What is the probability that a 4 is obtained both times? Why?

Solution: The repetitions of the experiment are independent so we may multiply probabilities. The probability of getting a 4 once is 3/36 = 1/12. Thus the probability of getting a 4 both times is (1/12)(1/12) = (1/144).

(3) Suppose the experiment is repeated 10 times. What is the probability that exactly 6 out of the 10 times result in a 7?

Solution: Let E be the event that exactly six out of ten tosses result in a 7. The number of ways of obtaining exactly six 7s out of ten tosses is "10 choose 6" = 210. Thus E contains 210 outcomes. The probability of a 7 is 1/6. The probability of getting something other than a 7 is 5/6. Thus each outcome in E has probability $(1/6)^6(5/6)^4 \approx .0000103$. The probability of E is the sum of the probabilities of the outcomes in E and so $P(E) = 210 * (.0000103) = .00217$.

(4) Suppose that the experiment is repeated 10 times. What is the probability that at least 2 out of the 10 times result in an odd number?

Solution: Let E be the event that at least 2 of the 10 tosses results in an odd number. The probability of getting an odd number from one repetition of the experiment is $P(3) + P(5) + P(7) + P(9) + P(11) = 1/2$. The probability of getting no odd numbers is

$(1/2)^{10} = .0009765625$. The probability of getting exactly one odd number is $10 * (1/2)^{10} = .009765625$. Thus the probability of not getting E is .0107421876. The probability of getting E is one minus this number. So $P(E) = .9892578125$.

(5) Suppose that the experiment is repeated 10 times and that at least 2 of the 10 times resulted in an odd number. What is the probability that exactly 6 out of the 10 times resulted in a 7?

Solution: Let F be the even that at least 2 of the 10 tosses is an odd number. Let E be the event that exactly 6 out of the 10 tosses results in a 7. We want $P(E|F)$. This is equal to $P(E \cap F)/P(F)$. From the last problem, we know that $P(F) = .9892578125$. The event $E \cap F$ is the event that at least 2 of the 10 tosses results in an odd number and that exactly 6 out of the 10 tosses results in a 7. Since 7 is an odd number, $E \cap F = E$. From above, we know that $P(E) = .00217$. Thus, $P(E|F) \approx .00219$ (which is slightly higher than P(E).)

(6) Suppose that Alf and Bettina are betting on the outcome of the experiment. If a number 2 - 6 is obtained, Bettina gives Alf \$10. If a number 8 - 12 is obtained, Alf gives Bettina \$10. If a 7 is rolled, they each give \$2 to charity. If the game is repeated many times, how much on average will Bettina win or lose?

Solution: Let A be the event that a 2 - 6 is obtained. Let B be the event that a 8 - 12 is obtained. Let $P(7)$ be the probability that a 7 is obtained. We have $P(A) = P(B) = 15/36$ and $P(7) = 6/36$. Bettina's expected value is:

$$(-\$10)P(A) + (\$10)P(B) + (-\$2)P(7) = \$(-150/36 + 150/36 - 12/36) = -\$1/3$$

On average, Bettina will lose 1/3 of a dollar.

(7) Suppose that Alf and Bettina have played the game (from the previous problem) 4 times and that Bettina has won exactly 3 out of the 4. What is the probability that if they play the game 2 more times (for a total of 6 times) that she would win at least 4 out of the 6 games?

Solution: Bettina needs at least one more win. This event is $\{BB, AB, BA, B7, 7B\}$. The probabilities are:

$$\begin{aligned} P(BB) &= (15/36)(15/36) \\ P(AB) &= (15/36)(15/36) \\ P(BA) &= (15/36)(15/36) \\ P(B7) &= (15/36)(6/36) \\ P(7B) &= (6/36)(15/36) \end{aligned}$$

The probability that Bettina wins is the sum of these:

$$(45/36)(15/36)+(12/36)(15/36)\approx .65972.$$

(8) Consider 2 events. Event E is the event that a regular Marijuana smoker uses Heroine at some point in their life. Event F is the event that someone who has used Heroine at some point in their life has also been a regular Marijuana smoker.

(a) If you want to figure out whether or not Marijuana is a “gateway drug”, should you calculate $P(E|F)$ or $P(F|E)$? Why?
Solution: $P(F|E)$. You want know how likely someone who smokes Marijuana is to go on to use Heroine.

(b) If you want to study the drug habits of Heroine users, should you calculate $P(E|F)$ or $P(F|E)$? Why?
Solution: $P(E|F)$. You are only concerned with the people who you know are using Heroine.

(c) Explain why the difference between $P(E|F)$ and $P(F|E)$ is important.
Solution: The probabilities can be very different and they capture different situations.