Spring 2014 MA 314: Exam 1

The Rules: You have from the time you receive the exam until 5 PM on Monday, March 7 to complete the exam and turn in your answer. You may use your textbooks and class notes, but you may not use any other resources in print or online. You may not talk about the exam with anyone except the professor.

Sign here after completing the exam to indicate that you have read and abided by the rules:

A word of advice: Start this early. Don't turn in your first drafts. Turn in beautiful, readable, clear work.

Do 4 out of the following 6 problems.

(1) Give a formula (in terms of a and b) for an isometry of $\mathbb{E}^2$ that takes the point (a,b) to the origin and which takes the vector $(\frac{\sqrt{2+\sqrt{2}}}{2}, \frac{\sqrt{2-\sqrt{2}}}{2})$ in the tangent space at (a,b) to the vector $(0,1)$ in the tangent space at the origin. (Hint: $(\cos(\pi/8), \sin(\pi/8)) = (\frac{\sqrt{2+\sqrt{2}}}{2}, \frac{\sqrt{2-\sqrt{2}}}{2})$).

(2) Show that there is a linear fractional transform $f(z) = \frac{az+b}{cz+d}$ whose restriction to $\mathbb{H}^2$ is an isometry and which takes any three given distinct points z_-, z_+, and z_0 in $\mathbb{R} \cup \{\infty\}$ to the points -1, $+1$, and ∞ (respectively) in $\mathbb{C} \cup \{\infty\}$. Use this to conclude that any two ideal triangles in $\mathbb{H}^2$ are isometric.

(3) Consider the cylinder of radius 1:

$$C = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$$

The length of a path on C is the same as its length thought of as a curve in $\mathbb{E}^3$.

Let γ: $[a,b] \to C$ be a C^1 path on C. If $\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ we can express γ in cylindrical coordinates as:

$$\widehat{\gamma}(t) = \begin{pmatrix} r(t)\cos\theta(t) \\ r(t)\sin\theta(t) \\ z(t) \end{pmatrix}$$

To calculate lengths in cylindrical coordinates, we need to know that:

$$\text{length}(\gamma) = \int_a^b \left(\sqrt{\dot{r}(t)^2 + r^2\dot{\theta}(t)^2 + \dot{z}(t)^2} \right) dt$$

(a) Let $P = (\cos\theta, \sin\theta, 0) \in C$ and $Q = (\cos\phi, \sin\phi, 0) \in C$ be points given in cylindrical coordinates. Find the length of the circular arc joining the points.

(b) Prove that any path in C joining P and Q (from the previous problem) has length at least the length of a circular arc joining them.

(c) Prove that any path in C joining points with the same angle θ is at least as long as the straight line joining them.

(d) (BONUS): Given $P = (a,b,z_0)$ and $Q = (c,d,z_1)$ on C describe the geodesics joining P to Q and prove that one of them is the shortest path (in C) between P and Q.

(4) Let $\mathbf{x}$ and $\mathbf{y}$ be two points in $\mathbb{E}^2$. Let

$$L = \{\mathbf{z} \in \mathbb{E}^2 : d(\mathbf{x},\mathbf{z}) = d(\mathbf{y},\mathbf{z})\}.$$

Prove that L is a line. (Here d is the euclidean metric. Hint: Use isometries rather than brute-force calculations.)

(5) Let $\sigma \in (0,\pi)$. Prove that there exists a triangle in $\mathbb{H}^2$ whose angle sum is exactly σ. (By triangle, I mean a geodesic triangle with all of its vertices non-ideal.)

(Hint: You may want to use the intermediate value theorem. For extra-credit prove that you can also specify the angles in advance – as long the angle sum is strictly less than π.)

(6) Suppose that $U \subset \mathbb{R}^2$ and $V \subset \mathbb{R}^2$ are both path-connected open sets and suppose that $h\colon\ U \to V$ is a differentiable function such that for each $p \in U$, $Dh|_p$ is an invertible matrix. (Recall that $Dh|_p$ is the matrix of partial derivatives of h at p.) The map h need not be an isometry.

(a) Let d_V be a metric on V. Define a function $d_U\colon\ U \times U \to \mathbb{R}$ by:

$$d_U(a,b) = d_V(h(a),h(b)).$$

Prove that if h is an injection, then d_U is a metric on U.

(b) Let $\langle\cdot,\cdot\rangle_y^V$ be a Riemannian metric on V. (The subscript y means that we plug into the inner product vectors in the tangent space based at y. The definition of Riemannian metric requires that the inner product is a smooth function of y.)

Prove that $\langle\cdot,\cdot\rangle^U$ defined by

$$\langle v,w\rangle_x^U = \langle Dh|_x v, Dh|_x w\rangle_{h(x)}^V$$

is a Riemannian metric on U. (Your answer should not depend on the fact that h is an injection, only on the assumption that $Dh|_p$ is invertible. You will need the fact from linear algebra that if a matrix is invertible then when you take it times a non-zero vector you get a non-zero vector.)

(This problem shows that differentiable maps can be used to create Riemannian metrics.)