# MA 398: The lesser path taken

Recall that $\mathbb{E}^2$ denotes $\mathbb{R}^2$ with the euclidean path distance:

$$l(\gamma) = \int_a^b ||\gamma'(t)||\,dt$$

where $\gamma\colon\ [a,b] \to \mathbb{R}^2$ is a piecewise $C^1$ path.

**Theorem 1.** Let $\gamma\colon\ [a,b] \to \mathbb{E}^2$ be a piecewise $C^1$ path. Then

$$l(\gamma) \geq ||\gamma(b) - \gamma(a)||.$$

Furthermore, if $\gamma$ is not a parameterization of the straight line, then the inequality is strict.

Notice that this implies that a straight line is the shortest path between two points in $\mathbb{E}^2$.

*Proof.* Let $(a_1, a_2) = \gamma(a)$ and $(b_1, b_2) = \gamma(b)$. Let $\gamma(t) = (x(t), y(t))$.

**Claim 1:** It is enough to prove the theorem for $C^1$ curves. We begin by showing that the theorem can be reduced to the case when $\gamma$ is $C^1$ instead of just piecewise $C^1$.

Since $\gamma$ is piecewise $C^1$, there exists $t_i$ so that:

$$a = t_0 < t_1 < \ldots < t_n = b.$$

On each interval $[t_i, t_{i+1}]$ the curve $\gamma$ is $C^1$, meaning that $x'$ and $y'$ are continuous and non-zero on these intervals. (At the points $t_i$, the derivatives $x'$ and $y'$ may not exist, but $x$ and $y$ have left hand and righthand continuous derivatives.) Let $\gamma_i$ denote the restriction of $\gamma$ to $[t_i, t_{i+1}]$.

By the triangle inequality, for any $i$

$$||\gamma(b) - \gamma(a)|| \leq \sum ||\gamma(t_{i+1}) - \gamma(t_i)||.$$

Also,

$$l(\gamma) = \sum l(\gamma_i).$$

Thus, if for all $i$,

$$||\gamma(t_{i+1}) - \gamma(t_i)|| \leq l(\gamma_i)$$

then $||\gamma(b) - \gamma(a)|| \leq l(\gamma)$. Thus, to prove the first part of the theorem, it suffices to prove it for the case when $\gamma$ is $C^1$.

Assume, for the moment, that each $\gamma_i$ is a parameterization of the line joining its endpoints. If $\gamma(t_i)$, $\gamma(t_{i+1})$, and $\gamma(t_{i+2})$ where not all collinear then the straight line joining $\gamma(t_i)$ to $\gamma(t_{i+2})$ would, by basic trigonometry, be a strictly shorter path than the path consisting of $\gamma$ restricted to the interval $[t_i, t_{i+1}]$. Thus, to prove the second part of the theorem, it also suffices to prove it for the case when $\gamma$ is $C^1$. □

**Claim 2:** It is enough to prove the theorem for the case when $x(a) = x(b) = 0$.

We know that rotation and translations are isometries for both the euclidean metric and the path metric on $\mathbb{R}^2$. There exists a line in $\mathbb{R}^2$ passing through the $\gamma(a)$ and $\gamma(b)$. Translate everything so that the line passes through the origin. Rotate everything so that the line coincides with the $y$-axis. The inverses of this rotation and translation are also a rotation and a translation. Since distances haven't changed if the theorem holds for the new points on the $y$-axis the theorem holds for the original points. □.

**Claim 3:** The theorem holds for $C^1$ curves joining two points on the $y$-axis.

Let $\gamma$ be $C^1$ path with $x(b) = x(a) = 0$. Since the square root function is a strictly increasing function, for a fixed $t$

$$||\gamma'(t)|| = \sqrt{(x'(t))^2 + (y'(t))^2} \geq \sqrt{(y'(t))^2} = |y'(t)|.$$

Notice also that equality holds if and only if $x'(t) = 0$.

If $f \leq g$ are integrable functions, then $\int f \leq \int g$, hence

$$\int_a^b ||\gamma'(t)||\, dt \geq \int_a^b |y'(t)|\, dt.$$

We also know that (assuming the integral exists) $|\int f| \leq \int |f|$, consequently,

$$l(\gamma) = \int_a^b ||\gamma'(t)||\, dt \geq \int_a^b |y'(t)|\, dt \geq |\int_a^b y'(t)|\, dt.$$

Since $y'$ is continuous, all those integrals exist and we can use the fundamental theorem of calculus to conclude that

$$l(\gamma) \geq |y(b) - y(a)| = ||\gamma(b) - \gamma(a)||,$$

since $x(b) = x(a) = 0$. We have, therefore, proven the first part of the theorem.

To prove the second part, we assume that $\gamma$ is not a parameterization of the straight line path.

**Claim:** The Theorem is true in the case when there exists $t_0 \in (a, b)$ such that $x(t_0) \neq 0$.

Since $x(a) = 0$, and since $x'$ is continuous, there exists $t_1 \in (a, t_0)$ such that $x'(t_1) \neq 0$. Since $(x')^2$ is continuous, there exists an $\varepsilon > 0$ such that $[t_1 - \varepsilon, t_1 + \varepsilon] \subset [a, t_0]$ and so that for all $t \in [t_1 - \varepsilon, t_1 + \varepsilon]$, $(x'(t))^2 > 0$. Since $(x')^2$ is continuous, there exists $m > 0$ such that for all $t \in [t_1 - \varepsilon, t_1 + \varepsilon]$, $x'(t) \geq m$. Let $\gamma_1$ be the restriction of $\gamma$ to $[a, t_1 - \varepsilon]$, $\gamma_2$ be the restriction of $\gamma$ to $[t_1 - \varepsilon, t_1 + \varepsilon]$, and $\gamma_3$ be the restriction of $\gamma$ to $[t_1 + \varepsilon, b]$.

Then,

$$l(\gamma) = l(\gamma_1) + l(\gamma_2) + l(\gamma_3).$$

By what we did earlier, we know that $l(\gamma_1) \geq ||\gamma_1(t_1 - \varepsilon) - \gamma_1(a)||$ and $l(\gamma_3) \geq ||\gamma_3(b) - \gamma_3(t_1 + \varepsilon)||$. Furthermore,

$$l(\gamma_2) \geq \int_{t_1 - \varepsilon}^{t_1 + \varepsilon} \sqrt{m + (y'(t))^2}\, dt.$$

Since $m > 0$, this last integral is strictly bigger than the integral of $\sqrt{(y'(t))^2} = |y'(t)|$. That integral is at least $\sqrt{m}(2\varepsilon)$. Thus, by the triangle inequality,

$$l(\gamma) \geq ||\gamma_1(t_1 - \varepsilon) - \gamma_1(a)|| + ||\gamma_3(b) - \gamma_3(t_1 + \varepsilon)|| + \sqrt{m}(2\varepsilon) \geq ||\gamma_3(b) - \gamma_1(a)|| + 2\varepsilon\sqrt{m}.$$

Since $\gamma_3(b) = \gamma(b)$ and $\gamma_1(a) = \gamma(a)$ and since $2\varepsilon\sqrt{m} > 0$, we have shown, as desired, that

$$l(\gamma) > ||\gamma(b) - \gamma(a)||.$$

□(Case)

Suppose, therefore, that $\gamma$ is a $C^1$ path such that $x(t) = 0$ for all $t$. Then $x'(t) = 0$ for all $t$ and so $y'(t) \neq 0$ for all $t$. Since $y'$ is continuous, either $y' > 0$ or $y' < 0$. In either case, we see that $y$ is a strictly monotonic function. It is, therefore, an injection and so $\gamma$ is an injection. Hence, $\gamma$ is the straight line path from $\gamma(a)$ to $\gamma(b)$. □

**Theorem 2.** Let $U \subset \mathbb{R}^n$ be path connected. Let $d(P, Q)$ for $P, Q \in U$ be defined by

$$d(P, Q) = \inf_{\gamma} l(\gamma)$$

where the infimum is taken over all piecewise $C^1$ functions joining $P$ to $Q$. Then $d$ is a metric.

*Proof.* We must prove that $d$ is positive definite, symmetric, and satisfies the triangle inequality. Recall that the length of a $C^1$ path is the integral of its speed and that the length of a piecewise $C^1$ path is the sum of the lengths of the $C^1$ pieces.

**Claim 1:** $d$ is positive definite.

Let $P, Q \in U$. Since $U$ is path-connected, there exists a piecewise $C^1$ path joining $P$ to $Q$. That path has non-infinite length and so $d(P, Q)$ is finite.

Since the length of a $C^1$ path $\gamma$ is the integral of $||\gamma'||$, it must be non-negative. The infimum of a set of non-negative is non-negative, so $d(P,Q) \geq 0$. The constant path $\gamma\colon [0,0] \to U$ defined by $\gamma(0) = P$ is $C^1$ (by definition) and so $d(P,P) = 0$. Other than the constant path any $C^1$ path has non-zero length and so (other than the constant path) every piecewise $C^1$ path has non-zero length. By the previous theorem, every path joining $P$ to $Q$ has length at least the length of the line segment joining $P$ to $Q$. That length is $||P-Q||$ which is non-zero if $P \neq Q$. Thus, $d(P,Q) \geq 0$. Hence, $d$ is positive definite.

**Claim 2:** $d$ is symmetric.

Let $P,Q \in U$ and let $\gamma\colon [a,b] \to U$ be a piecewise $C^1$ path joining $P$ to $Q$. Define $\zeta : [-b,-a] \to U$ by

$$\zeta(t) = \gamma(-t).$$

Notice that $||\zeta'(t)|| = ||\gamma'(t)||$ whenever $\gamma'$ exists. Thus, $\zeta$ is piecewise $C^1$. Furthermore,

$$\int_{-b}^{-a} ||\zeta'(t)||\,dt = -\int_{-b}^{-a} ||\gamma'(-t)||(-1)\,dt = \int_{a}^{b} ||\gamma'(t)||\,dt$$

by substitution. Thus, the length of $\gamma$ and $\zeta$ are the same. Thus, in the definitions of $d(P,Q)$ and $d(Q,P)$ we are taking the infima over the same set of lengths and so $d(P,Q) = d(Q,P)$.

**Claim 3:** $d$ satisfies the triangle inequality

Let $P,Q,R \in U$. Let $\varepsilon > 0$. Since $d(P,Q)$ is the infimum of lengths of all piecewise $C^1$ paths in $U$ joining $P$ to $Q$, there exists a piecewise $C^1$ path $\gamma_{PQ}$ joining $P$ to $Q$ with length $l(\gamma_{PQ}) \leq d(P,Q) + \varepsilon/2$. Similarly, there exists a piecewise $C^1$ path $\gamma_{QR}$ joining $Q$ to $R$ with length $l(\gamma_{QR}) \leq d(Q,R) + \varepsilon/2$. The path $\zeta$ which travels along $\gamma_{PQ}$ and then along $\gamma_{QR}$ is a piecewise $C^1$ path joining $Q$ to $R$. It has length

Since for all $\varepsilon$, $d(P,R) \leq l(\zeta)$ we have that for all $\varepsilon > 0$:

$$d(P,R) \leq d(P,Q) + d(Q,R) + \varepsilon.$$

This is true for all $\varepsilon > 0$ and so $d(P,R) \leq d(P,Q) + d(Q,R)$. □