# MA 398 Homework 8: Have you found inner product?

## 1. Reading

Read Section 1.4 of Bonahon (again). Read Section T.4 of Bonahon.

## 2. Huts

(1) Prove that the set of isometries of $\mathbb{E}^3$ fixing the origin is a subset of the isometries of $S^2 = \{\mathbf{x} : ||\mathbf{x}|| = 1\}$.

(2) Prove that if $\phi$ is an isometry of $S^2$ then there is a unique isometry $\widehat{\phi}$ of $\mathbb{E}^3$ such that the restriction of $\widehat{\phi}$ to $S^2$ is equal to $\phi$.

(3) Use the previous two problems to show that the set of isometries of $S^2$ is equal to $O(3)$ (the set of linear maps of $\mathbb{R}^3$ whose transpose is the inverse.) (Hint: Appeal to some results from class.)

## 3. Houses

Recall that an inner product $\langle \cdot, \cdot \rangle$ on a (finite dimensional, real) vector space $V$ is a symmetric bilinear map from $V \times V$ to $\mathbb{R}$. The standard example is the dot product on $\mathbb{R}^n$. Given an inner product we can define a norm on $V$ by $||x|| = \langle x, x \rangle^{1/2}$ and a metric by $d(x,y) = ||x - y||$. For the following problems, you may assume that $V = \mathbb{R}^n$ and that the inner product is the standard dot product.

The goal of these exercises is to prove that any isometry of $V$ fixing the origin is an orthogonal linear map (in the sense of linear algebra).

Suppose that $\phi$ is an isometry of $V$. That is, for all $v, w \in V$ we have $d(v,w) = d(\phi(v), \phi(w))$. Also assume that $\phi(\mathbf{0}) = \mathbf{0}$.

(1) Prove that if $v \in V$, then $||\phi(v)|| = ||v||$. In particular, $\langle \phi(v), \phi(v) \rangle = \langle v, v \rangle$.

(2) For $a, b \in V$, use the fact that $d^2(a,b) = d^2(\phi(a), \phi(b))$ to show that $\langle a, b \rangle = \langle \phi(a), \phi(b) \rangle$. (Hint: use the fact that
$$\langle a - b, a - b \rangle = ||a||^2 - 2\langle a, b \rangle + ||b||^2.)$$
As a result of this exercise, we say that "$\phi$ preserves inner product".

(3) Let $(e_1, \dots, e_n)$ be an orthonormal basis for $V$. (That is, each $e_i$ has magnitude 1 and $\langle e_i, e_j \rangle$ equals 0 if $i \neq j$ and 1 if $i = j$.) For $a \in V$, write $a = \sum_{i=1}^n a_i e_i$ for $a_i \in \mathbb{R}$. Prove that if for all $a$

$$\phi(a) = \sum_{i=1}^n a_i \phi(e_i)$$

then $\phi$ is linear.

(4) Prove that for all $a \in V$, $\phi(a) = \sum_{i=1}^n a_i \phi(e_i)$. Conclude that $\phi$ is a linear map. Since $\phi$ is a linear map that preserves the inner product it is orthogonal.

Hint: Let $b = \sum_{i=1}^n a_i \phi(e_i)$ and use the bilinearity of the inner product to show that

$$0 = ||a - b||^2 = \langle a - b, a - b \rangle.$$

## 4. Cathedrals

In this section you'll investigate the properties of differentials of isometries. These results follow most easily from the classification of isometries of $\mathbb{R}^2$, but you needn't appeal to that result to prove these. In fact, these results hold for Riemannian metric.

Let $U \subset \mathbb{R}^2$ and $V$ be open subsets and for $p \in U \cup V$, let $T_p$ denote the tangent space to $p$ with standard basis vectors $(e_1, e_2)$. Recall that if $\phi\colon\ U \to V$ is differentiable, then for each $p \in U$ there is a map

$$L\colon\ T_p \to T_{\phi(p)}$$

defined by

$$L(v) = D\phi|_p v$$

where $D\phi|_p$ is the derivative of $\phi$ evaluated at $p$.

(1) Suppose that $\phi\colon\ U \to V$ is a differentiable bijection with inverse $\psi$. Use the multivariable chain rule to prove that for all $p$

$$(D\phi|_p)^{-1} = D\psi|_{\phi(p)}.$$

(2) Suppose that $\phi\colon\ U \to V$ is a differentiable bijective isometry. Prove that for each $p \in U$, the determinant of $D\phi|_p$ is $\pm 1$. (Hint: Use the fact that there is a $\psi\colon\ V \to U$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are each the identitities and that determinant is a multiplicative function.)