**MA 398 Homework 4: This one is integral to the course.**

Our focus is shifting from the category of all metric spaces to euclidean geometry. We've seen two metrics on subsets of euclidean space: the euclidean metric and the path metric. These metrics give the answer when we consider all of $\mathbb{R}^n$ but may give different answers on proper subsets of $\mathbb{R}^n$. The reading below shows how the path metric can be used to give an intrinsic geometry to the 2-dimensional sphere $S^2 \subset \mathbb{R}^3$. The Houses section finishes up the details of showing that on all of $\mathbb{R}^2$ the path metric and the euclidean metric give the same answer. It also gives you the opportunity to think more about the basic isometries of $\mathbb{R}^2$. You'll need to do some calculus, but hopefully you think that's fun!

## 1. Reading

- Schwartz: Section 9.1 and 9.2
- Bonahon: Chapter 3.

## 2. Huts

These problems are intended to give you some practice with basic concepts. They will often involve calculation, rarely involve new ideas, and won't be graded. However, your answers will be collected!

(1) Give 3 (very different) examples of a set $U \subset \mathbb{R}^2$ and points $x, y \in U$ such that if $d$ is the path metric then $d(x,y)$ is not realized by a path joining $x$ to $y$. (Recall that the path metric is the infimum of the lengths of all (piecewise $C^1$) paths in $U$ joining $x$ to $y$. So you are looking for examples where the infimum is not obtained by any piecewise $C^1$ path)

(2) Do exercises 2 and 6 on page 33 of Schwartz. (You don't need to be completely rigorous in Exercise 6. For now, just think of "homeomorphic" as meaning "can be continuously deformed to look like".)

## 3. Houses

These problems are intended to require more thought and less calculation.

(1) Do exercise 1.2 on page 8 of Bonahon.

(2) Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be a $C^1$ path. Let $\phi: \mathbb{R}^2 \to \mathbb{R}^2$ be a translation, rotation about the origin, or reflection across the $x$-axis. Prove (using the definitions) that the length of $\gamma$ is equal to the length of $\phi \circ \gamma$. (You have to consider the 3 possibilities separately). Show how to use this to conclude that these translations, rotations, and reflections are isometries of $\mathbb{R}^2$ using the path metric.

## 4. Cathedrals

These problems show how to extend the notion of path length to metric spaces where we can't take derivatives.

(1) Do exercise 1.11 on page 9 of Bonahon.
(2) Do exercise 1.12 on page 9 of Bonahon.