MA 398 Homework 2: What Sup? and What Inf?

In class we defined infimum and supremum of subsets of $\mathbb{R}$. The first few problems are intended to reinforce those ideas. The other problems on this homework will get you ready for discussing continuous functions.

1. Reading

- Schwartz: Sections 2.3 and 2.4
- Bonahon: Section 1.3 and appendix T.2

Remark: Take the time to verify that Schwartz and Bonahon are giving some of the same definitions. Try drawing some pictures of sequences in metric spaces (like subsets of $\mathbb{R}^2$) and figuring out what their limits are.

2. Huts

These problems are intended to give you some practice with basic concepts. They will often involve calculation, rarely involve new ideas, and won't be graded. However, your answers will be collected!

(1) Find the sup and inf of the following subsets of $\mathbb{R}$. Also, say whether or not the sup and inf are elements of the set or not.

  (a) $(0,1)$ (this is the interval consisting of precisely those $x \in \mathbb{R}$ such that $0 < x < 1$)

  (b) $[0,1)$ (this is the interval consisting of precisely those $x \in \mathbb{R}$ such that $0 \leq x < 1$)

  (c) $(0,1] \cup [2,3)$

  (d) $\{7\}$

  (e) $\{\frac{1}{n} : n \in \mathbb{N}\}$. (Recall that $\mathbb{N} = \{1,2,3,\ldots\}$ is the set of natural numbers.)

  (f) $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$.

(2) Give an example of two sequences (x_n) and (y_n) of real numbers such that for all n, $x_n \neq y_n$, but

$$\inf_{n \in \mathbb{N}} \{x_n - y_n\} = 0.$$

3. HOUSES

These problems are intended to require more thought and less calculation.

(1) Let (X,d) be a metric space. A sequence (x_n) in X has the **limit** $x \in X$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x) < \varepsilon$. We also say that (x_n) **converges** to x.

(a) Let (x_n) be a constant sequence. That is, there exists $x \in X$ such that for all n, $x_n = x$. Prove that (x_n) converges to x.

(b) Suppose that (x_n) converges to $x \in X$. Prove that the limit is unique. That is, prove that if (x_n) also converges to $y \in X$, then $x = y$. (Hint: Do a proof by contradiction.)

(c) Let (X,d) be $\mathbb{R}$ with the usual metric. Suppose that (x_n) is a bounded, increasing sequence. (That is, for all n, $x_{n+1} \geq x_n$) Prove that there exists $x \in \mathbb{R}$ such that (x_n) converges to x. (Hint: Prove that (x_n) converges to the supremum of $\{x_n \in \mathbb{R} : n \in \mathbb{N}\}$).

4. CATHEDRALS

Problems in this section require significant effort or imagination. They are worth a relatively small part of the grade.

(1) Let d_1 be the usual metric on $\mathbb{R}^2$ and let d_2 be the sup metric. Prove the following:

(a) If B_1 is an open ball in $\mathbb{R}^2$ defined using d_1 and if $x \in B_1$, then there exists an open ball B_2 defined using d_2 such that $x \in B_2$ and $B_2 \subset B_1$. (Hint: draw a picture of what open balls look like in each metric.)

(b) If B_2 is an open ball in $\mathbb{R}^2$ defined using d_2 and if $x \in B_2$, then there exists an open ball B_1 defined using d_1 such that $x \in B_1$ and $B_1 \subset B_2$.

(c) Define $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x) = x$. Give the domain of f the metric d_1 and give the codomain the metric d_2. Prove that f is a homeomorphism. (That is, prove that f is a bijection, f is continuous, and f^{-1} is continuous.) This problem requires more sophistication that the previous two parts and the challenge is probably to write something coherent. The first two parts of the problem are definitely relevant.