MA 398 Homework 16: Do you believe in the Truth Farey?

1. HUTS

(1) Read Bonahon chapter 8.
(2) Read Schwartz chapter 19.

2. HUTS

(1) Summarize the relationship between the Farey Graph, a hyperbolic structure on the punctured torus, and the Ford circle packing.
(2) Summarize the relationship between continued fractions and the Farey Graph.

3. HOUSES

(1) Do Exercise 6 of Schwartz. (page 245)

(2) (Bonus!) Do Exercise 7 of Schwartz. This exercise shows that $PSL_2\mathbb{Z}$ is isomorphic to the free product of $\mathbb{Z}/2/\mathbb{Z}$ with $\mathbb{Z}/3\mathbb{Z}$.

(3) Let T be the flat torus obtained by gluing opposite sides of the rectangle $[0,1]\times[0,1]$ without a twist. Let $\mathscr{P}$ be the tiling of $\mathbb{E}^2$ by squares arisng from T.

   (a) Prove that for every rational number $p/q \in \mathbb{Q}\cup\{1/0\}$ a line $l(p/q)$ of slope p/q in $\mathbb{E}^2$ descends to a geodesic $g(p/q)$ in T that is a closed, non-self intersecting, closed loop.

   (b) Prove that $g(p/q)$ and $g(p'/q')$ intersect exactly once if and only if $pq' - qp' = \pm 1$. (This problem is possibly very challenging. If you get stuck, just take it as given and move on to the next one.)

   (c) Form a graph G as follows. For each $g(p/q)$ take a vertex $v(p/q)$. Join $v(p/q)$ and $v(p'/q')$ by an edge if and only if $g(p/q)$ and $g(p'/q')$ intersect exactly once on T. Explain why this graph is, in some sense, the "same" as the Farey graph.

   (d) Let $h\colon T \to T$ be a homeomorphism. Notice that if α and β are curves that intersect once on T then $h(\alpha)$ and $h(\beta)$ are as well. It is also a fact that if h and h' are homeomorphisms of T to itself such that if there exist curves α and β on T intersecting once with $h(\alpha) = h'(\alpha)$ and $h(\beta) = h'(\beta)$ then (up to composing h or h' with an orientation reversing homeomorphism fixing α or β) the homeomorphism h can be "deformed" into the homeomorphism h'.

Do your best to explain why these facts mean that the group of homeomorphisms of T (up to "deformation") is a subgroup of the symmetry group of the Farey graph. (Feel free to be informal, or to add conditions or hypotheses as need.)

The point is that the homeomorphisms of the torus are closely related to the symmetries of the Farey graph which are in turn closely related to hyperbolic geometry (via the modular group). This is supposed to give you some idea why it's not outlandish to suppose that the space of all euclidean structures on the torus can be described using hyperbolic geometry.