MA 398 Homework 16: Do you believe in the Truth Farey?

1. HUTS

(1) Read Bonahon chapter 8.
(2) Read Schwartz chapter 19.

2. HUTS

(1) Summarize the relationship between the Farey Graph, a hyperbolic structure on the punctured torus, and the Ford circle packing.
(2) Summarize the relationship between continued fractions and the Farey Graph.

3. HOUSES

(1) Do Exercise 6 of Schwartz. (page 245)

(2) (Bonus!) Do Exercise 7 of Schwartz. This exercise shows that $PSL_2\mathbb{Z}$ is isomorphic to the free product of $\mathbb{Z}/2/\mathbb{Z}$ with $\mathbb{Z}/3\mathbb{Z}$.

(3) Let T be the flat torus obtained by gluing opposite sides of the rectangle $[0,1]\times[0,1]$ without a twist. Let $\mathscr{P}$ be the tiling of $\mathbb{E}^2$ by squares arisng from T.

(a) Prove that for every rational number $p/q \in \mathbb{Q}\cup\{1/0\}$ a line $l(p/q)$ of slope p/q in $\mathbb{E}^2$ descends to a geodesic $g(p/q)$ in T that is a closed, non-self intersecting, closed loop.

Solution: We know from previous homework that $g(p/q)$ is a geodesic. We need to show that it is closed and non-self intersecting. Let $\pi\colon \mathbb{R}^2 \to T$ be the quotient map. Let $X \subset \mathbb{R}^2$ be a square whose edges are glued to obtain T. Let $x \in X$ be a point in $g(p/q)$. Thinking of $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ as a vector we see that $y = \begin{pmatrix} x_1+q \\ x_2+p \end{pmatrix}$ is a point in $\mathbb{R}^2$ obtained by shifting x to the right by q and up by p. Thus $x \sim y$ and so $\pi(x) = \pi(y)$. Also y is on the line $l(p/q)$ since we've moved x up by p and over by q. Thus the line segment in $l(p/q)$ descends to T as a closed loop. The gluing map glues $l(p/q)$ to all sets obtained by translating the points of $l(p/q)$ by vectors with integer entries. The result of doing that to $l(p/q)$ is a line parallel to $l(p/q)$. That line either coincides with $l(p/q)$ or is disjoint from $l(p/q)$. Thus, $g(p/q)$ does not intersect itself.

(b) Prove that $g(p/q)$ and $g(p'/q')$ intersect exactly once if and only if $pq' - qp' = \pm 1$. (This problem is possibly very challenging. If you get stuck, just take it as given and move on to the next one.)

Solution: Let $g = g(p/q)$ and $g' = g(p'/q')$ and let l and l' be the corresponding lines in $\mathbb{R}^2$. Suppose, first, that g and g' intersect exactly once. Cutting T open along g and g' produces a disc. The lines l and l' and their images under the group action give a tiling of $\mathbb{R}^2$ by parallelograms. The edges of the parallelograms glue up to become g and g'. Choose a coordinate system on $\mathbb{R}^2$ so that l and l' intersect at the origin and let v and v' be the edges of the parallelogram with endpoints at the origin. Since T was formed by gluing opposite edges of the square, it has area 1. Thus the parallelogram defined by v and v' also has area 1. Thinking of v and v' as vectors we recall that parallelogram spanned by them has area 1 if and only if $|v \times v'| = 1$. We have $v = (p,q)$ and $v' = (p',q')$ and so $|v \times v'| = |pq' - qp'|$. Thus, if g and g' intersect exactly once, $pq' - qp' = \pm 1$.

Suppose now that $pq' - qp' = 1$. Without loss of generality we may assume that $l = l(p/q)$ and $l' = l(p',q')$ go through the origin. Let A be the matrix:

$$A = \begin{pmatrix} p & p' \\ q & q' \end{pmatrix}.$$

Notice that $x \to Ax$ takes the standard basis vectors of $\mathbb{R}^2$ onto the lines l and l' and that, by assumption, A has determinant 1. The transformation

$$\alpha(x) = A^{-1}x = \begin{pmatrix} q' & -p' \\ -q & p \end{pmatrix} x$$

therefore takes the line l and l' to the coordinate axes of $\mathbb{R}^2$. Furthermore, $\beta = \pi \circ \alpha \circ \pi^{-1}$ is a homeomorphism of the torus since A^{-1} and A have integer entries. (You should think about that!) Since β is a homeomorphism, the number of times that $\beta(g)$ and $\beta(g')$ intersect is the same as the number of times that g and g' intersect. But $\beta(g) = g(0/1)$ and $\beta(g') = g(1/0)$ and these intersect exactly once, so we are done.

(c) Form a graph G as follows. For each $g(p/q)$ take a vertex $v(p/q)$. Join $v(p/q)$ and $v(p'/q')$ by an edge if and only if $g(p/q)$ and $g(p'/q')$ intersect exactly once on T. Explain why this graph is, in some sense, the "same" as the Farey graph.

Solution: The Farey graph has vertices in one-to-one correspondence with rational numbers$\{p/q\} \cup \{1/0\}$ (with p/q reduced). Two vertices v and v' corresponding to rationals p/q and p'/q' are joined by an edge if and only if $pq' - qp' = \backslash 1$. The function that takes the line $l(p/q)$ in $\mathbb{R}^2$ (with p/q reduced) to the vertex in the Farey graph corresponding to the rational p/q is a graph isomorphism because two lines intersect once if and only if the corresponding vertices in the Farey graph are joined by an edge.

(d) Let $h\colon\ T \to T$ be a homeomorphism. Notice that if α and β are curves that intersect once on T then $h(\alpha)$ and $h(\beta)$ are as well. It is also a fact that if h and h' are homeomorphisms of T to itself such that if there exist curves α and β on T intersecting once with $h(\alpha) = h'(\alpha)$ and $h(\beta) = h'(\beta)$ then (up to composing h or h' with an orientation reversing homeomorphism fixing α or β) the homeomorphism h can be "deformed" into the homeomorphism h'.

Do your best to explain why these facts mean that the group of homeomorphisms of T (up to "deformation") is a subgroup of the symmetry group of the Farey graph. (Feel free to be informal, or to add conditions or hypotheses as need.)

The point is that the homeomorphisms of the torus are closely related to the symmetries of the Farey graph which are in turn closely related to hyperbolic geometry (via the modular group). This is supposed to give you some idea why it's not outlandish to suppose that the space of all euclidean structures on the torus can be described using hyperbolic geometry.

Solution: A given curve α in T can be deformed so as to intersect the meridian $g(1/0)$ and the longitude $g(0/1)$ minimally some number of times each. There is also a geodesic $g(p/q)$ for some p/q intersecting the meridian and longitude the same number of times and α can be deformed to $g(p/q)$. In fact, there are a whole family of geodesics, each parallel to $g(p/q)$ with this property. We associate α to the corresponding rational p/q.

Homeomorphisms of T preserve the number of points of intersection between curves on T and geodesics realize the minimal number of intersections between curves that can be deformed to them. So beginning with two geodesics g and g' that intersect once, applying a homeomorphism $h\colon\ T \to T$ sends them to curves α and β that intersect once. The curves α and β can be deformed to geodesics k and k' that intersect once. Using the correspondence with the Farey graph we see that h takes two vertices of the Farey graph connected by an edge in the Farey graph to two other vertices connected by the Farey graph. Thus, h gives a graph isomorphism of the Farey graph to itself. We know that the symmetries of the Farey graph are described (up to some minor details) by the tiling group associated to a complete hyperbolic structure on the punctured torus which is itself a subgroup of $PSL_2(\mathbb{R})$. Thus the group of homeomorphisms of the torus (up to some minor details) is related to a subgroup of the group of hyperbolic isometries.