MA 398 Homework 11: Thin as a triangle

This homework is due on Fri. March 16. It is recommended that you do or attempt the Huts and Houses (1) and (2) by Wed.

1. Huts

(1) Schwartz, Exercise 6, page 123. You can follow his hints or use Bonahon, Proposition 2.15 (which we discussed in class). You may also find it useful to remember that angles are preserved by an isometry ϕ if and only if for every point $p \in \mathbb{H}^2$ and all vectors v, w in the tangent space at p, $\langle v, w \rangle_p = \langle D\phi|_p v, D\phi|_p w \rangle_{\phi(p)}$. (Here $\langle \cdot, \cdot \rangle$ represents the hyperbolic Riemannian metric.).

(2) In class and in Bonahon, Prop. 2.15 we proved the result only for linear fractional maps. Write down the argument for anti-linear fractional maps.

(3) Find a hyperbolic isometry that takes the vector $(0,1)$ at the point $i \in \mathbb{C}$ to the vector $(1,0)$ at the point $i \in \mathbb{C}$.

2. Houses

(1) Do exercises 2.6 and 2.9 of Bonahon.

(2) Do parts a and b of Bonahon exercise 2.14. Read parts c and d.

(3) Do exercises 2.16 and 2.17 of Bonahon.

3. Cathedrals

(1) Recall that in class as part of the theorem classifying hyperbolic isometries we showed that if τ is a hyperbolic isometry that fixes the positive imaginary axis L and if g_y is the complete geodesic through yi for $y > 0$ and perpendicular to L then either for all $z \in g_y$, $\tau(z) = z$ or $\tau(z) = -\bar{z}$. That is, τ either fixes g_y or flips g_y. The point of this problem is to show that for all $z \in \mathbb{H}$, either $\tau(z) = z$ or $\tau(z) = -\bar{z}$. To that end:

Suppose that $\tau \colon \mathbb{H}^2$ is a continuous map from $\mathbb{H} \to \mathbb{H}$ such that for each g_y the map τ either flips g_y or fixes g_y.

(a) Let y be a positive real number and suppose that y_n is a sequence of positive reals converging to y. Using the continuity of τ, explain why if τ flips each g_{y_n} then τ also flips g_y and if τ fixes each g_{y_n} then τ fixes g_y. (Hint: recall that if f is a continuous function and if (x_n) is a sequence converging to x, then the sequence $(f(x_n))$ converges to $f(x)$.)

(b) Let $\varepsilon_0 > 0$. Suppose that y is a positive real number such that if n is large enough, then there exists y_n such that $y - 1/n \leq y \leq y_n < y + 1/n$ and $\tau(y_n)$ flips g_{y_n}. Prove that τ flips g_y. Explain why the same result holds if we replace "flips" with "fixes".

(c) Let S be the set of positive y-values such that τ flips g_y. Prove that if S is non-empty then S consists of all positive y-values. (Hint: Assume that S is non-empty and show that the inf of S is 0 and the sup of S is ∞. Do this using a proof by contradiction and the previous parts of the problem.)

(2) Do problem 2.13 of Bonahon.