MA 398 Homework 1: Metric Spaces go the distance!

Metric spaces will be the foundation of all of our geometric work this semester. They are intended to axiomatize the idea of "space with a distance".

1. Reading

- Schwartz: Sections 2.1 and 2.2
- Bonahon: Preface and Section 1.3

Remark: For the time being you should pay attention mostly to the axioms for metric spaces. We'll worry about the other definitions later. Notice that Bonahon lists 4 axioms but Schwartz lists only 3 – how do you reconcile that?

What Bonahon calls a *semi-metric* we will sometimes call a *pseudo-metric*. Our own Ben Mathes has a particular fondness for things called "partial pseudo-metrics". We won't discuss those though...

2. Huts

These problems are intended to give you some practice with basic concepts. They will often involve calculation, rarely involve new ideas, and won't be graded. However, your answers will be collected!

Show that the following are metric spaces:

(1) The real numbers $\mathbb{R}$ with metric d defined by $d(x,y) = |x-y|$.

(2) Any set X with a metric $d\colon X \times X \to \mathbb{R}$ defined by:

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

This metric is called the *discrete metric*

(3) Suppose that (X,d) is a metric space and that $Y \subset X$. Define $d\big|_Y\colon Y \times Y \to \mathbb{R}$ by

$$d\big|_Y(x,y) = d(x,y)$$

Prove that $d\big|_Y$ is a metric. It is called the *restriction* of d to Y.

(4) Suppose that (X,d_X) and (Y,d_Y) are metric spaces. Recall that the *product* of X and Y is the set:

$$X \times Y = \{(x,y) : x \in X \text{ and } y \in Y\}$$

Define metrics d_1 and d_2 on $X \times Y$ by

$$d_1\Big((x_1,y_1),(x_2,y_2)\Big) = d_X(x_1,x_2) + d_Y(y_1,y_2)$$

and

$$d_2\Big((x_1,y_1),(x_2,y_2)\Big) = \max\Big(d_X(x_1,x_2), d_Y(y_1,y_2)\Big).$$

Prove that d_1 and d_2 are metrics. The first metric is called the "taxi-cab metric" and the second metric is called the "sup" metric.

3. Houses

These problems are intended to require more thought and less calculation.

(1) Do exercise 2 on page 23 of Schwartz's book. Our own Fernando Gouvêa has written an entire book about such metric spaces! We won't see them again, though.

(2) Give an example of a set X and a function $d\colon X \times X \to \mathbb{R}$ satisfying the following:

(a) $d(x,y) = 0$ if and only if $x = y$.
(b) For all $x,y,z \in X$,

$$d(x,y) + d(y,z) \geq d(x,z).$$

But such that there exist $x,y \in X$ such that $d(x,y) \neq d(y,x)$.

Hint: You can do this with a space having exactly two points.

4. Cathedrals

Problems in this section require significant effort or imagination. They are worth a relatively small part of the grade.

These problems all concern isometries. An *isometry* of a metric space (X,d) is a function $f\colon X \to X$ such that for all $x,y \in X$,

$$d(f(x),f(y)) = d(x,y).$$

(1) Give two examples of an isometry of $\mathbb{R}$ (with the usual metric).

(2) Prove that an isometry is always injective.

(3) Give an example of a metric space (X,d) and an isometry of X that is not surjective.

(4) Suppose that (X,d) is a metric space. Let $\mathscr{I}$ be the set of isometries of X. Prove:

(a) For all $f,g \in \mathscr{I}$ the function $f \circ g \in \mathscr{I}$.
(b) There is an identity element in $\mathscr{I}$. That is, there exists an isometry ϕ of X such that for all $f \in \mathscr{I}$

$$\phi \circ f = f \text{ and } f \circ \phi = f.$$

(c) Suppose that $f \in \mathscr{I}$. Then there exists $g \in \mathscr{I}$ such that

$$f \circ g = \phi \text{ and } g \circ f = \phi$$

if and only if f is surjective.

The previous exercise shows that if X has the property that every isometry is surjective, then the set of isometries is a "group". We'll work more with groups later in the semester. Roughly speaking, they are algebraic ways of keeping track of the symmetries of a space.

(5) Give an example of a metric space such that the set of isometries is finite. Can you give an example where the metric space has infinitely many points?