MA 398: The Grasshopper Metric

Let $X \subset \mathbb{E}^2$ be a polygon consisting of finitely many regions bounded by finitely many line segments, half lines, or lines. Let the edges of X be

$$E_1, E_2, \ldots, E_{2m}$$

Let ϕ_i be an isometry from edge E_{2i} to edge E_{2i-1} for $1 \leq i \leq m$ and let $\overline{X}$ be the result of gluing edge E_{2i} to edge E_{2i-1} using ϕ_i. (This is formalized using an equivalence relation.) For each $P \in X$, let $\overline{P} \in \overline{X}$ be the equivalence class of P, that is: the set of points glued to P, including P itself. Notice that:

- If P is in the interior of X, then $\overline{P} = \{P\}$.
- If P is on an edge but is not an endpoint, then $\overline{P}$ consists of two points P, and one other point that is either $\phi_i(P)$ or $\phi_i^{-1}(P)$ for some i.
- If P is an endpoint (henceforth, called a vertex) of an edge, then $\overline{P}$ consists finitely many points, all of which are vertices of X.

We assume that X is convex. (If it is not, then we can subdivide it so that it is.) Then euclidean distance d is a path metric on X. Recall that for $\overline{P}, \overline{Q} \in \overline{X}$ we define

$$\delta(\overline{P}, \overline{Q}) = \min d(P, Q)$$

where the minimum is taken over all $P \in \overline{P}$ and $Q \in \overline{Q}$. Since X has finitely many edges, there are only finitely many such P and Q.

Recall that a chain $\overline{C}$ in $\overline{X}$ from $\overline{P}$ to $\overline{Q}$ is a finite sequence of points in $\overline{Q}$:

$$C : \overline{P} = \overline{P}_0, \overline{P}_1, \ldots, \overline{P}_n = \overline{Q}$$

The length of $\overline{C}$ is

$$\delta(\overline{C}) = \sum \delta(\overline{P}_i, \overline{P}_{i+1}).$$

Given such a chain, there is an associated virtual chain C in X:

$$C : Q_0, P_1, Q_1, P_2, Q_2, P_2, \ldots, Q_{n-1}, P_n$$

such that for each i, $\overline{P}_i = \overline{Q}_i$ and $\delta(\overline{P}_i, \overline{P}_{i+1}) = d(Q_i, P_{i+1})$. We think about the chain C as consisting of a sequence of drives and teleports. We drive from Q_0 to P_1, teleport to Q_1, drive to P_2, teleport to Q_2, etc. Notice that:

$$\delta(\overline{C}) = \sum d(Q_i, P_{i+1}).$$

For $\overline{P}, \overline{Q} \in \overline{X}$, define the grasshopper metric $\overline{d}$ by

$$\overline{d}(\overline{P}, \overline{Q}) = \inf \delta(\overline{C})$$

where the infimum is taken over all chains $\overline{C}$ in $\overline{X}$ joining $\overline{P}$ to $\overline{Q}$. Let $\overline{B}_\varepsilon(\overline{P})$ denote all points $\overline{Q} \in \overline{X}$ such that $\overline{d}(\overline{P}, \overline{Q}) < \varepsilon$.

Let $\overline{P} \in \overline{X}$ and define $\varepsilon_0(\overline{P})$ as follows:

- If $P \in \overline{P}$ is in the interior of X, let ε_0 be the minimum distance between P and the points on the edges of X.
- If $P \in \overline{P}$ is in an edge of X, let ε_0 be the minimum distance between P and the points on the edges of X not containing P. Take the minimum over all $P \in \overline{P}$.

Theorem 1. Suppose that X is the union of convex polygons with edge gluings $\{\phi_i\}$. and that $\overline{X}$ is connected. Let $\overline{P} \in \overline{X}$ and let $0 < \varepsilon \leq \varepsilon_0(\overline{P})$. Let $\pi\colon X \to \overline{X}$ be the quotient map. Then the following are true:

(1) For all $\overline{Q} \in \overline{B}_\varepsilon(\overline{P})$ there exist $P \in \overline{P}$ and $Q \in \overline{Q}$ such that $\overline{d}(\overline{P}, \overline{Q}) = d(P, Q)$.
(2) $\pi^{-1}(\overline{B}_\varepsilon(\overline{P})$ is the disjoint union of $B_\varepsilon(P)$ where the union is taken over all $P \in \overline{P}$.

Proof. Let $\rho > 0$ be such that $\varepsilon + \rho < \varepsilon_0$. and choose a chain $\overline{C}$ from $\overline{P}$ to $\overline{Q}$ having $\delta(\overline{C}) < \overline{d}(\overline{P}, \overline{C}) + \rho$. Out of all such

$$\overline{C} : \overline{P}_0, \overline{P}_1, \ldots, \overline{P}_n$$

choose $\overline{C}$ so as to minimize n. We claim that $n = 1$. To see this, let

$$C : Q_0, P_1, Q_1, P_2, \ldots, Q_{n-1}, P_n$$

be the associated chain in Q.

Case 1: $P_1 = Q_1$

In this case we have, by the triangle inequality, $d(Q_0, P_2) \leq d(Q_0, Q_1) + d(Q_1, Q_2)$. The chain

$$C' : Q_0, P_2, Q_2, \ldots, Q_{n-1}, P_n$$

therefore has fewer terms and

$$\overline{C}' : \overline{P}_0, \overline{P}_2, \ldots, \overline{P}_n$$

has $\delta(\overline{C}') \leq \delta(\overline{C}) < d(\overline{P}, \overline{Q}) + \rho$ and has fewer terms that $\overline{C}$. This contradicts our choice of $\overline{C}$.

Case 2: P_1 and Q_1 lie on edges of X.

By the definition of the gluing, there exists a isometry ϕ_i from any edge containing P_1 to an edge containing Q_1 or vice versa. Without loss of generality, assume $\phi_i(P_1) = Q_1$. Since $d(Q_0, P_1) \leq \overline{d}(\overline{P}, \overline{Q}) + \rho < \varepsilon_0$, the point Q_0 must lie on an edge of X containing P_1. Let ϕ_i be the isometry such that $\phi_i(P_1) = Q_1$ and ϕ_i is defined on Q_0. Then because ϕ_i is an isometry

$$d(Q_0, P_1) + d(Q_1, P_2) = d(\phi_i(Q_0), Q_1) + d(Q_1, P_2) \leq d(\phi_i(Q_0), P_2).$$

Hence, the chain

$$C' : \phi_i(Q_0), P_2, Q_2, \ldots, Q_{n-1}, P_n$$

is a chain with fewer terms then C and its quotient chain

$$\overline{C}' : \overline{P}_0, \overline{P}_2, \ldots, \overline{P}_n$$

is still a chain from $\overline{P}$ to $\overline{Q}$ with $\delta(\overline{C}') < d(\overline{P}, \overline{Q}) + \rho$. This contradicts our choice of $\overline{C}$. $\square$

Notice that the previous result shows that there do not exist distinct points $\overline{P}, \overline{Q} \in \overline{X}$ with $\overline{d}(\overline{P}, \overline{Q}) = 0$. Hence, the grasshopper metric is a metric.

We now improve this to show

Theorem 2. Let $\overline{P} \in \overline{X}$ and let $\varepsilon < \varepsilon_0/3$. Then for each $P \in \overline{P}$, the restricted map

$$\pi \colon B_\varepsilon(P) \to \overline{B}_\varepsilon(\overline{P})$$

is an isometry (not necessarily surjective).

Proof. Let $Q \in B_\varepsilon(P)$. Notice also that if $P \in \overline{P}$ is not on an edge of X, then Q is not on an edge of X. If P is on an edge of X, then either Q is not on an edge of X or it is on an edge that contains P.

Let v be a point on an edge of X not containing P. Suppose that $d(Q, v) < 2\varepsilon/3$.Then,

$$d(P, v) \leq d(P, Q) + d(Q, v) < \varepsilon/3 + 2\varepsilon/3 = \varepsilon.$$

But $d(P, v) \geq \varepsilon_0(\overline{P}) > \varepsilon$ by the definition of ε. Hence, $2\varepsilon/3 \leq \varepsilon_0(Q)$. By the previous theorem, this implies that if $Q' \in B_{2\varepsilon/3}(Q)$, then $d(Q, Q') = \overline{d}(\overline{Q}, \overline{Q}')$.

Now suppose that $Q, Q' \in B_\varepsilon(P)$. By the triangle inequality, $d(Q, Q') \leq d(Q, P) + d(P, Q) < 2\varepsilon/3$. Consequently, $d(Q, Q') = \overline{d}(\overline{Q}, \overline{Q}')$. $\square$