MA 398: Exam 3

The Rules: The exam will be administered during exam period 4 on Thursday May 10 from 9 AM to 12 PM. You may not use textbooks, notes, electronic devices, or other people while writing your answers to the exam. The exam has been distributed in advance – you are encouraged to spend time thinking about your answers and discussing them with other people prior to when the exam is given.

Your exams will be graded on completeness, correctness, and clarity. You are encouraged to tell me more than the bare minimum, but that does not mean that you should incorporate irrelevant material.

Main Task: Give a thorough explanation and proof that all compact surfaces without boundary can be given a metric that makes them locally isometric to 2-dimensional euclidean, spherical, or hyperbolic space. Furthermore, prove that such a surface is uniquely euclidean, spherical, or hyperbolic. (That is, if the surface admits a euclidean structure, it does not admit a spherical or hyperbolic one, etc. It is not the case that the euclidean (spherical, hyperbolic) structure is unique.)

You may assume without definition or proof the following:

- the definition of metric space.
- calculus
- the definition of infimum and supremum
- the definition of surface
- basic topological notions such as continuity, euler characteristic, and the topological classification of surfaces.
- the definition of equivalence relation, equivalence classes, and quotient sets.
- the basic geometric properties of the euclidean, spherical, and hyperbolic plane. (i.e. Geodesics, isometries, isotropy, etc.)
- the Gauss-Bonnet theorem

You may also assume that the reader has a fair bit of mathematical sophistication, but you may not assume that they have encountered this result before.

The following are questions you are encouraged to provide answers to as part of your answer to the Main Task. You do not necessarily need to answer them in the order listed or to answer the question exactly the way it is stated.

(1) Suppose that (X,d) is a metric space and that $\sim$ is an equivalence relation on X. Let $\overline{X}$ be the set of equivalence classes (the quotient set).

  (a) Define *chain* and *grasshopper metric*

  (b) Prove that the grasshopper metric is a pseudo-metric.

(2) Describe the geodesics and isometries in $\mathbb{E}^2$, S^2, and $\mathbb{H}^2$. You do not need to prove that these are the geodesics and isometries.

(3) Define *euclidean polygon*, *spherical polygon*, and *hyperbolic polygon* and define the notion of edge gluing maps.

(4) For a euclidean, spherical, or hyperbolic polygon X, define conditions on X and on the edge gluing maps that guarantee that $\overline{X}$ is surface, that the grasshopper metric $\overline{d}$ is a metric (and not just a pseudo-metric), and that $(\overline{X},\overline{d})$ is locally isometric to $\mathbb{E}^2$, S^2, or $\mathbb{H}^2$.

(5) Prove your statement from the previous problem for the case when X is a euclidean polygon.

(6) Discuss what changes, if any, need to be made to your proof for euclidean polygons to adapt it to work for spherical and hyperbolic polygons.

(7) Explain the statement of the topological classification theorem for surfaces and why it implies that every compact surface without boundary can be obtained by gluing edges of regular polygon.

(8) Suppose that $\overline{X}$ is a compact surface without boundary obtained by gluing edges of a regular polygon X. Explain why X can be embedded in $\mathbb{E}^2$, S^2, or $\mathbb{H}^2$ so that it and its gluing maps satisfy the conditions of (4).

(9) The Gauss-Bonnet theorem says:

Theorem 1 (Gauss-Bonnet)**.**

$$2\pi\chi(\overline{X}) = K\text{Area}(X)$$

where X is a polygon in $\mathbb{E}^2$, S^2, or $\mathbb{H}^2$ with gluing maps satisfying (4) and $\chi(\overline{X})$ is the euler characteristic of $\overline{X}$. The constant K is defined to be 0 if X is a polygon in $\mathbb{E}^2$, 1 if X is a polygon in S^2, and -1 if X is a polygon in $\mathbb{H}^2$.

Explain why this theorem (together with the topological classification of surfaces) implies that if a compact surface $\overline{X}$ without boundary admits a metric that is locally isometric to $\mathbb{E}^2$ then it is the torus or Klein bottle and if $\overline{X}$ admits a metric locally isometric to S^2 then it is either S^2 or the projective plane P^2.