MA 398: Exam 2

The Rules: You have from the time you receive the exam until 5 PM on Tuesday, April 17 to complete the exam and turn in your answer. You may use your textbooks and class notes, but you may not use any other resources in print or online. You may not communicate about the exam with anyone except the professor.

Sign here after completing the exam to indicate that you have read and abided by the rules:

A word of advice: Start this early. Don't turn in your first drafts. Turn in beautiful, readable, clear work.

1. Calculations and Equations

Do one of the following problems.

(1) Give an example of a polygon X in each of $\mathbb{E}^2$, $\mathbb{H}^2$, and S^2 and isometric gluing maps of the edges such that $\overline{X}$ is a Möbius band. You should specify exactly what the gluing maps of the edges are (in terms of the elementary isometries of each space) and exactly what the polygon X is. You will probably want $\overline{X}$ to be an "open" Möbius band – i.e. a Möbius band without boundary. See Bonahon Section 5.4.2 for hints. (You may like to note that the "spherical" Möbius band will necessarily be incomplete.

(2) A pair of pants is a surface with boundary homeomorphic to the result of removing three open discs from S^2. Give a proof, including as many details as you can, that a pair of pants can be given a geometry that is locally isometric to $\mathbb{H}^2$ (for interior points) or to $\{z \in \mathbb{H}^2 : (Re)(z) \geq 0\}$. (You may reference any theorems from the book or class unless they make this problem completely trivial.)

2. Theorems and Proofs

Do both of the following:

(1) Let U be any isotropic path metric space (such as, but not limited to, $\mathbb{E}^2$, $\mathbb{H}^2$, or S^2). Suppose that $X \subset U$ is a polygon and that E is an edge of X. Let X' be another such polygon and E' an edge of X'. Suppose that $\phi: E \to E'$ is an isometry of the edges. Prove that there is an isometry $\psi: U \to U$ such that the restriction of ψ to E is ϕ and and so that $\psi(X)$ is on the same side of E' as X'. (Hint: Consider Bonahon Lemma 4.8). A counter example to the claim that we can also guarantee $\psi(X) = X'$ will give you bonus points.

(2) Let $X \subset \mathbb{H}^2$ be a closed and bounded polygon and let $\{\phi_i\}$ be isometries between pairs of edges of X, as usual. Let $\overline{X}$ be the resulting surface. Assume that X satisfies the hypotheses of Bonahon Theorem 4.10 so that $\overline{X}$ is a surface without boundary locally isometric to $\mathbb{H}^2$.

   (a) Appeal to results from class/Bonahon to show that the gluing maps $\{\phi\}$ give rise to a tiling $\mathscr{P}$ of $\mathbb{H}^2$ by tiles each isometric to X.

   (b) Define a function $\pi: \mathbb{H}^2 \to \overline{X}$ such that the restriction of π to each tile of $\mathscr{P}$ gives the associated quotient map to $\overline{X}$.

   (c) Suppose that $g \subset \overline{X}$ is a curve. Prove that g is a geodesic if and only if there is a geodesic $\widetilde{g} \subset \mathbb{H}^2$ such that $\pi \circ \widetilde{g} = g$.

   (d) Give an example to show that the previous result does not hold if $\mathbb{H}^2$ is replaced by X.

   (e) (Challenge!) Suppose that $f: \overline{X} \to \overline{X}$ is an isometry. Prove that there is an isometry $\widetilde{f}: \mathbb{H}^2 \to \mathbb{H}^2$ such that for all $x \in \mathbb{H}^2$,

$$\pi \circ \widetilde{f}(x) = f \circ \pi(x).$$