MA 398: Exam 1

The Rules: You have from the time you receive the exam until 5 PM on Tuesday, March 6 to complete the exam and turn in your answer. You may use your textbooks and class notes, but you may not use any other resources in print or online. You may not talk about the exam with anyone except the professor.

Sign here after completing the exam to indicate that you have read and abided by the rules:

A word of advice: Start this early. Don't turn in your first drafts. Turn in beautiful, readable, clear work.

1. Calculations and Equations

Do two of the following three problems:

(1) Give a formula (in terms of a and b) for an isometry of $\mathbb{E}^2$ that takes the point (a,b) to the origin and which takes the vector $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ in the tangent space at (a,b) to the vector $(0,1)$ in the tangent space at the origin.

(2) Suppose that $\gamma\colon\ [a,b] \to \mathbb{H}^2$. The **energy** of γ is given by:

$$E(\gamma) = \int_a^b \frac{||\gamma'(t)||^2}{y^2}\, dt$$

where y is the y-coordinate of γ. Calculate the energy of the curve

$$\gamma(t) = \begin{pmatrix} t \\ e^{-t} \end{pmatrix}$$

for $0 \leq t \leq 7$.

(3) Consider the cylinder of radius 1:

$$C = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$$

The length of a path on C is the same as its length thought of as a curve in $\mathbb{E}^3$.

We can use cylindrical coordinates to describe points on C. The formulas for converting from cylindrical coordinates to Cartesian coordinates are:

$$\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \\ z &= z. \end{aligned}$$

The angle θ is the same angle as in polar coordinates (the angle between (x,y) and the positive x axis.)

Let $\gamma\colon\ [a,b] \to C$ be a C^1 path on C.

(a) Find a formula for $||\gamma'(t)||$ in cylindrical coordinates.

(b) Find the length of a path in C of your choosing joining the point $(-1,0,0)$ to $(0,1,1)$.

2. Theorems and Proofs

Do two of the following:

(1) Let $\mathbf{x}$ and $\mathbf{y}$ be two points in $\mathbb{E}^2$. Let
$$L = \{\mathbf{z} \in \mathbb{E}^2 : d(\mathbf{x}, \mathbf{z}) = d(\mathbf{y}, \mathbf{z})\}.$$
Prove that L is a line. (Here d is the euclidean metric.)

(2) Suppose that P is a polygon in $\mathbb{E}^2$ and that pairs of edges are glued together by isometries to form $\overline{P}$. Give $\overline{P}$ the grasshopper metric. Let $a, b \in P$ be points such that the distance between a and b is strictly less than the distance from a to any edge of P and is also strictly less than the distance between b and any edge of P. Let $\overline{a}$ and $\overline{b}$ be the corresponding points in $\overline{P}$. Prove that the distance between $\overline{a}$ and $\overline{b}$ in $\overline{P}$ (with the grasshopper metric) is equal to the distance from a to b in P (with the euclidean metric.)

(3) Suppose that $U \subset \mathbb{R}^2$ and $V \subset \mathbb{R}^2$ are both path-connected open sets and suppose that $h\colon U \to V$ is a differentiable function such that for each $p \in U$, $Dh|_p$ is an invertible matrix. (Recall that $Dh|_p$ is the matrix of partial derivatives of h at p.) The map h need not be an isometry.

(a) Let d_V be a metric on V. Define a function $d_U\colon U \times U \to \mathbb{R}$ by:

$$d_U(a,b) = d_V(h(a),h(b)).$$

Prove that if h is an injection, then d_U is a metric on U.

(b) Let $\langle \cdot,\cdot \rangle_y^V$ be a Riemannian metric on V. (The subscript y means that we plug into the inner product vectors in the tangent space based at y. The definition of Riemannian metric requires that the inner product is a smooth function of y.)

Prove that $\langle \cdot,\cdot \rangle^U$ defined by

$$\langle v,w \rangle_x^U = \langle Dh|_x v, Dh|_x w \rangle_{h(x)}^V$$

is a Riemannian metric on U. (Your answer should not depend on the fact that h is an injection, only on the assumption that $Dh|_p$ is invertible. You will need the fact from linear algebra that if a matrix is invertible then when you take it times a non-zero vector you get a non-zero vector.)

(4) Suppose that $\gamma\colon [a,b] \to X$ is a path where (X,d) is a metric space. Define a **directed chain** in γ to be a finite sequence

$$\gamma(t_0), \gamma(t_1), \gamma(t_2), \ldots, \gamma(t_n)$$

such that

$$a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b.$$

Define the length of a chain C to be

$$L(C) = \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)).$$

Define the length of γ to be:

$$L(\gamma) = \sup_C L(C)$$

where C is a directed chain in γ. The point of this problem is to show that this definition of length coincides with the definition of length for C^1 curves in $\mathbb{E}^2$.

Let (X,d) be $\mathbb{E}^2$, i.e. $\mathbb{R}^2$ with the euclidean metric.

(a) Explain why the euclidean length of a C^1 curve $\gamma \subset \mathbb{R}^2$ is no smaller than the length of any chain lying in γ. (You may appeal to previously proven results.)

(b) Let γ be a C^1 curve in $\mathbb{R}^2$ and let C_n be the chain in γ such that $t_i - t_{i-1} = (b-a)/n$ for all i. Prove that $\lim_{n\to\infty} L(C_n)$ is the euclidean length of γ.

Hint: You may like to use the following two results from Calculus:

Theorem (Mean Value Theorem)**.** Suppose that $f\colon [\alpha,\beta] \to \mathbb{R}$ is continuous and differentiable on (α,β). Then there exists $c \in (\alpha,\beta)$ such that

$$f(\beta) - f(\alpha) = (\beta - \alpha) f'(c).$$

Theorem (Riemann Integration)**.** Suppose that $g\colon [a,b] \to \mathbb{R}$ is continuous and that for $n \geq 1$ we have

$$a = t_0 < t_1 < t_2 < \ldots < t_n = b.$$

so that for all i, $t_i - t_{i-1} = \frac{b-a}{n}$. Choose $t_i^* \in [t_{i-1}, t_i]$. Then

$$\lim_{n\to\infty} \sum_{i=1}^{n} g(t_i^*) \frac{(b-a)}{n} = \int_a^b g\,dx.$$