MA 262: Practice Exam 1 Solutions **Name:**_________________________

This practice exam is much longer than the actual exam.

(1) Let $F(x,y) = (x^2y, y^2x, 3x - 2yx)$. Find the derivative of F.

Solution:

$$DF(x,y) = \begin{pmatrix} 2xy & x^2 \\ y^2 & 2yx \\ 3-2y & -2x \end{pmatrix}$$

(2) Let $F(x,y) = (x-y, x+y)$ and let $G(x,y) = (x\cos y, x\sin y)$. Find the derivative of $F \circ G$ using the chain rule.

Solution:

$$\begin{aligned} DF(x,y) &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ DG(x,y) &= \begin{pmatrix} \cos y & -x\sin y \\ \sin y & x\cos y \end{pmatrix} \\ D(F\circ G)(x,y) &= DF(G(x,y))DG(x,y) \\ &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \cos y & -x\sin y \\ \sin y & x\cos y \end{pmatrix} \\ &= \begin{pmatrix} \cos y - \sin y & -x\sin y - x\cos y \\ \cos y + \sin y & -x\sin y + x\cos y \end{pmatrix} \end{aligned}$$

(3) Suppose that a rotating circle of radius 1 is travelling through the plane, so that at time t seconds the center of the circle is at the point $(t, \sin t)$. Let P be the point on the circle which is at $(0,1)$ at time $t = 0$. If the circle makes 3 revolutions per second, what is the path $\mathbf{x}(t)$ taken by the point P?

Solution: The rotation of the P relative to the center of the circle (that is, in $T_{\mathbf{c}(t)}$) can be described by the path $(\cos(6\pi t + \pi/2), \sin(6\pi t + \pi/2))$. Thus, $\mathbf{x}(t) = (\cos(6\pi t + \pi/2) + t, \sin(6\pi t + \pi/2) + \sin t)$.

(4) A rotating circle of radius 1 follows a helical path in $\mathbb{R}^3$ so that at time t the center of the circle is at $(\sin t, \cos t, t)$. At each time t,

the circle lies in the osculating plane. (That is, the circle lies in the plane spanned by the unit tangent and the unit normal vectors.) Let P be the point on the circle which is at $(1,0)$ at time $t = 0$. The circle completes one rotation every 2π seconds. Find a formula $\mathbf{x}(t)$ for the path taken by the point P.

(5) A rotating circle of radius 1 follows a helical path in $\mathbb{R}^3$ so that at time t the center of the circle is at $(\sin t, \cos t, t)$. At each time t, the circle lies in the osculating plane. (That is, the circle lies in the plane spanned by the unit tangent and the unit normal vectors.) Let P be the point on the circle which is at $(1,0)$ at time $t = 0$. The circle completes one rotation every 2π seconds. Find a formula $\mathbf{x}(t)$ for the path taken by the point P. (Hint: Express the center of the circle as a combination of the unit tangent and normal vectors.)

Solution: Relative to the center of the circle (that is, in $T_{\mathbf{c}(t)}$) the point P follows the path $\cos t\mathbf{T} + \sin t\mathbf{N}$ where $\mathbf{T}$ and $\mathbf{N}$ are the unit tangent and unit normal vectors to $\mathbf{c}(t) = (\sin t, \cos t, t)$ respectively. Those formulae are

$$\begin{aligned} \mathbf{T}(t) &= \tfrac{1}{\sqrt{2}} \begin{pmatrix} \cos t \\ -\sin t \\ 1 \end{pmatrix} \\ \mathbf{N}(t) &= \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix} \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{c}(t) &= \cos t\mathbf{T} + \sin t\mathbf{N} + \mathbf{c}(t) \\ &= \tfrac{\cos t}{\sqrt{2}} \begin{pmatrix} \cos t \\ -\sin t \\ 1 \end{pmatrix} + \sin t \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix} + \begin{pmatrix} \sin t \\ \cos t \\ t \end{pmatrix}. \end{aligned}$$

(6) Explain what it means for curvature to be an intrinsic quantity.

Solution: The curvature of a path $\mathbf{x}(t)$ at t_0, depends only on the curve itself at t_0, not on the parameterization $\mathbf{x}$.

(7) Prove that the curvature at any point of a circle of radius r is $1/r$.

Solution: A circle of radius r can be parameterized as $\mathbf{x}(t) = (r\cos t, r\sin t)$ for $0 \leq t \leq 2\pi$. We have:

$$\begin{array}{rcl} \mathbf{x}'(t) & = & (-r\sin t, r\cos t) \\ ||\mathbf{x}'(t)|| & = & r \\ \mathbf{T} & = & (-\sin t, \cos t) \\ \mathbf{T}' & = & (-\cos t, -\sin t) \\ ||\mathbf{T}'|| & = & 1 \\ \kappa(t) & = & ||\mathbf{T}'||/||\mathbf{x}'|| \\ & = & 1/r. \end{array}$$

(8) Let $\mathbf{x}(t) = (\cos t, \sin t, t)$ for $1 \leq t \leq 2$. Find $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ (that is, the moving frame) for $\mathbf{x}$ and also find κ (the curvature).

Solution: We have:

$$\begin{array}{rcl} \mathbf{x}'(t) & = & (-\sin t, \cos t, t) \\ ||\mathbf{x}'(t)|| & = & \sqrt{2} \\ \mathbf{T}(t) & = & \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1) \\ \mathbf{T}'(t) & = & \frac{1}{\sqrt{2}}(-\cos t, -\sin t, 0) \\ \mathbf{N}(t) & = & (-\cos t, -\sin t, 0) \\ \mathbf{B}(t) & = & \frac{1}{\sqrt{2}}(\sin t, -\cos t, 1) \\ \kappa(t) & = & 1/2 \end{array}$$

(9) Suppose that $\mathbf{x}\colon [a,b] \to \mathbb{R}^n$ is a $\mathbf{C}^1$ path such that for all t, $||\mathbf{x}(t)|| = 5$. Prove that at each t, $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ are perpendicular.

Solution: Since $5 = \mathbf{x}(t) \cdot \mathbf{x}(t)$, by the product rule, we have $0 = 2\mathbf{x} \cdot \mathbf{x}'$, implying that $\mathbf{x}$ and $\mathbf{x}'$ are perpendicular.

(10) A particle is following the path $\mathbf{x}(t) = (t, t^2, t^3)$ for $1 \leq t \leq 5$. Find an integral representing the distance travelled by the particle after t seconds.

Solution: The distance travelled after t seconds is

$$\begin{array}{rcl} s(t) & = & \int_1^t ||\mathbf{x}'(\tau)||\, d\tau \\ & = & \int_1^t \sqrt{1 + 4t^2 + 9t^4}\, d\tau. \end{array}$$

(11) Let $\mathbf{x}(t) = (t^2, 3t^2)$ for $t \geq 1$. Reparameterize $\mathbf{x}$ by arc length.

Solution: We compute,

$$s(t) = \int_1^t \sqrt{4\tau^2 + 36\tau^2}\, d\tau = \int_1^t 2\tau\sqrt{10}\, d\tau. = \sqrt{10}(t^2 - 1).$$

Then,

$$s^{-1}(t) = \sqrt{t/\sqrt{10} + 1}$$

Consequently,

$$\mathbf{y}(t) = \mathbf{x} \circ s^{-1}(t) = (t/\sqrt{10} + 1, 3t/\sqrt{10} + 3)$$

is the reparameterization of $\mathbf{x}$ by arclength.

(12) Suppose that $\mathbf{x}(t)$ is a path in $\mathbb{R}^n$ such that $\mathbf{x}(0) = \mathbf{a}$ and $\mathbf{x}(1) = \mathbf{b}$ (that is, $\mathbf{x}$ is a path joining $\mathbf{a}$ to $\mathbf{b}$.) Find a path which has the same image as $\mathbf{x}$ but which joins $\mathbf{b}$ to $\mathbf{a}$.

Solution: $\mathbf{y}\colon [-1, 0] \to \mathbb{R}^n$ defined by $\mathbf{y}(t) = \mathbf{x}(-t)$ will do the trick since $\mathbf{y}(0) = \mathbf{a}$ and $\mathbf{y}(-1) = \mathbf{b}$.

(13) Let $\mathbf{x}\colon [a, b] \to \mathbb{R}^n$ be a path with $\mathbf{x}'(t) \neq \mathbf{0}$ for all t. Let $\mathbf{y} = \mathbf{x} \circ \phi$ be an orientation reversing reparameterization of $\mathbf{x}$. Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is integrable. Prove that $\int_{\mathbf{y}} f\, ds = \int_{\mathbf{x}} f\, ds$.

Solution: Since ϕ is orientation reversing, $|\phi'(t)| = -\phi'(t)$. Hence, $||\mathbf{y}'(t)|| = -||\mathbf{x}'(\phi(t))||\phi'(t)$. Thus,

$$\int_{\mathbf{y}} f\, ds = -\int_c^d f(\mathbf{x}(\phi(t)))||\mathbf{x}'(\phi(t))||\phi'(t)\, dt.$$

Substitute $u = \phi(t)$ and $du = \phi'(t)dt$ to get:

$$\int_{\mathbf{y}} f\, ds = -\int_b^a f(\mathbf{x}(u))||\mathbf{x}'(u)||\, du.$$

Reversing the limits of integration eliminates the negative sign and so the result follows.

(14) Let $\mathbf{x}(t) = (t\cos t, t\sin t)$ for $0 \leq t \leq 2\pi$. Let $f(x, y) = y\cos x$. Let $F(x, y) = (-y, x)$. Find one-variable integrals representing $\int_{\mathbf{x}} f\, ds$ and $\int_{\mathbf{x}} F \cdot d\mathbf{s}$.

Solution: Notice that

$$\begin{aligned}
\mathbf{x}(t) &= (t\cos t, t\sin t) \\
\mathbf{x}'(t) &= (\cos t - t\sin t, t\cos t + \sin t) \\
||\mathbf{x}'(t)|| &= \sqrt{(\cos t - t\sin t)^2 + (t\cos t + \sin t)^2}
\end{aligned}$$

Thus,

$$\int_{\mathbf{x}} f\, ds = \int_0^{2\pi} t\sin t\cos(t\cos t)\sqrt{(\cos t - t\sin t)^2 + (t\cos t + \sin t)^2}\, dt.$$

And,

$$\begin{aligned}
\int_{\mathbf{x}} F \cdot ds &= \int_0^{2\pi} \begin{pmatrix} -t\sin t \\ t\cos t \end{pmatrix} \cdot \begin{pmatrix} \cos t - t\sin t \\ \sin t + t\cos t \end{pmatrix} dt. \\
&= \int_0^{2\pi} t^2\, dt \\
&= 8\pi^3/3.
\end{aligned}$$

(15) The gravitation vector field in $\mathbb{R}^3$ is $F(\mathbf{x}) = -\mathbf{x}/||\mathbf{x}||^3$. Find an integral representing the amount of work done by gravity as an object moves through the vector field F along the path $\mathbf{x}(t) = (t\cos t, t\sin t, t)$ for $1 \leq t \leq 2\pi$.

Solution: Notice that:

$$\begin{aligned}
||\mathbf{x}(t)|| &= t\sqrt{2} \\
\mathbf{x}'(t) &= (\cos t - t\sin t, \sin t + t\cos t, 1)
\end{aligned}$$

Plugging the path into the vector field we get:

$$\mathbf{F}(\mathbf{x}(t)) = \frac{-1}{(t\sqrt{2})^3} \begin{pmatrix} t\cos t \\ t\sin t \\ t \end{pmatrix}.$$

The work done by $\mathbf{F}$ along $\mathbf{x}$ is equal to $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$. Thus, the work done by gravity is equal to:

$$\int_1^{2\pi} \frac{-1}{(t\sqrt{2})^3} \begin{pmatrix} t\cos t \\ t\sin t \\ t \end{pmatrix} \cdot \begin{pmatrix} \cos t - t\sin t \\ \sin t + t\cos t \\ 1 \end{pmatrix} dt.$$

We can write this as follows, to avoid scaring Calc I students:

$$\int_1^{2\pi} \frac{-1}{(t\sqrt{2})^3} \Big((t\cos t)(\cos t - t\sin t) + (t\sin t)(\sin t + t\cos t) + t \Big)\, dt.$$

(16) Let $\mathbf{F}(x, y) = (x, -2y)$.

(a) Sketch a portion of the vector field F.

(b) Sketch a flow line for the vector field starting at $(1, 1)$.

(c) Find a parameterization for the flow line starting at $(1, 1)$.

Solution: Let $\phi(t) = (x(t), y(t))$ be the flow line. Then we are looking for x and y so that:

$$\begin{aligned} x'(t) &= x(t) \\ y'(t) &= -2y(t) \\ x(0) &= 1 \\ y(0) &= 1 \end{aligned}$$

Using $x(t) = e^t$ and $y(t) = e^{-2t}$ does the trick.

(d) The vector field F is a gradient field. Find the potential function.

Solution: $f(x, y) = x^2/2 - y^2$.

(17) Let $F(x, y) = (2xy, x^2 + 1)$. Find a potential function for F.

Solution: $f(x, y) = x^2y + y$.

(18) Explain why flow lines for an everywhere non-zero gradient field never close up. Use this to prove that $\mathbf{F}(x, y) = (-y, x)$ is not a gradient field.

Solution: Let $\mathbf{F} = \nabla f$ be a continuous gradient field. As we travel along a flow line, the value of f is strictly increasing and so we cannot arrive back at the same point. The given vector field has $(\cos t, \sin t)$ for $0 \leq t \leq 2\pi$ as a flow line. This flow line closes up and so $\mathbf{F}$ cannot be a gradient field.

(19) Let $f(x, y) = ye^x$. Find the gradient of f.

Solution: $\nabla f(x, y) = (ye^x, e^x)$.

(20) Let $F(x, y, 0) = (ye^x, xe^{y^2}, 0)$. Find the divergence of F.

Solution: $\operatorname{div} f(x, y) = ye^x + 2yxe^{y^2}$

(21) Let $F(x, y, z) = (xyz, xe^y \ln(z), x^2 + y^2 + z^2)$. Find the curl of F.

(22) Find the curl of your answer to problem 16.

Solution:0

(23) Find the divergence of your answer to problem 18.

Solution: 0.

(24) Let $\mathbf{F}$ be a C^1 vector field. State the integral definition of the scalar curl of $\mathbf{F}$ at a point $\mathbf{a}$ and prove that it gives the same answer as the derivative definition for vector fields of the form $\mathbf{F} = (M, 0)$.

Solution: Let C_n be a rectangle around the point $\mathbf{a}$ so that as $n \to \infty$ all the points on C_n converge to $\mathbf{a}$. Orient C_n counterclockwise. Let A_n be the area enclosed by C_n. Then the scalar curl of $\mathbf{F}$ is defined to be

$$\lim_{n\to\infty} \frac{1}{A_n} \int_{C_n} \mathbf{F} \cdot d\mathbf{s}.$$

To prove that it gives the same answer as the derivative definition:

Let $C_n = [a_n, b_n] \times [c_n, d_n]$. Then

$$\frac{1}{A_n} \int_{C_n} \mathbf{F} \cdot d\mathbf{s} = \frac{1}{(b_n - a_n)(d_n - c_n)} \int_{a_n}^{b_n} M(t, c_n) - M(t, d_n)\, dt$$

since $\mathbf{F}$ is perpendicular to the vertical sides of C_n and since we can paramterize the sides of C_n by (t, c_n) and (t, d_n).

By the Mean Value Theorem for Integrals, there exists $x_0 \in [a_n, b_n]$ such that

$$M(x_0, c_n) - M(x_0, d_n) = \frac{1}{(b_n - a_n)} \int_{a_n}^{b_n} M(t, c_n) - M(t, d_n)\, dt.$$

Thus,

$$\frac{1}{(b_n-a_n)(d_n-c_n)} \int_{a_n}^{b_n} M(t, c_n) - M(t, d_n)\, dt = \frac{M(x_0,c_n)-M(x_0,d_n)}{d_n-c_n} = -\frac{M(x_0,d_n)-M(x_0,c_n)}{d_n-c_n}.$$

By the Mean Value Theorem for derivatives, there exists $y_0 \in [c_n, d_n]$ such that

$$\frac{\partial M}{\partial y}(x_0, y_0) = \frac{M(x_0, d_n) - M(x_0, c_n)}{d_n - c_n}.$$

Consequently,

$$\frac{1}{A_n} \int_{C_n} \mathbf{F} \cdot d\mathbf{s} = -\frac{\partial M}{\partial y}(x_0, y_0).$$

As $n \to \infty$, the point (x_0, y_0) which is inside C_n goes to $\mathbf{a}$. Since the partial derivatives of M ar continuous, we have

$$\lim_{n\to\infty} \frac{1}{A_n} \int_{C_n} \mathbf{F} \cdot d\mathbf{s} = -\frac{\partial M}{\partial y}(\mathbf{a}).$$

which is what we were trying to prove.

(25) Suppose that $\mathbf{F} = \nabla f$ is the gradient field of a C^1 scalar field f on $\mathbb{R}^2$. Let $\mathbf{x}\colon [a, b] \to \mathbb{R}^2$ be a C^1 path. State and prove the fundamental theorem of calculus for conservative vector fields.

Answer: The Fundamental Theorem of Calculus for Conservative Vector Fields says that if $\mathbf{F} = \nabla f$ is a conservative vector field, with f a C^1 vector field and if $\mathbf{x}\colon [a,b] \to \mathbb{R}^n$ is a C^1 path, then

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{x}(b)) - f(\mathbf{x}(a)).$$

To prove it, consider:

$$\frac{d}{dt} f(\mathbf{x}(t)) = Df(\mathbf{x}(t))\mathbf{x}'(t)$$

by the chain rule. Since Df is a row vector which is the transpose of $\nabla f = \mathbf{F}$ we can rewrite this as:

$$\frac{d}{dt} f(\mathbf{x}(t)) = \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t).$$

Integrate both sides:

$$\int_a^b \frac{d}{dt} f(\mathbf{x}(t))\, dt = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt.$$

By the Fundamental Theorem of Calculus, the left side is equal to $f(\mathbf{x}(b)) - f(\mathbf{x}(a))$ and the right side is equal to $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$ by the definition of the line integral. Thus,

$$f(\mathbf{x}(b)) - f(\mathbf{x}(a)) = \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}.$$