

This practice exam is much longer than the actual exam.

- (1) Let $F(x, y) = (x^2y, y^2x, 3x - 2yx)$. Find the derivative of F .
- (2) Let $F(x, y) = (x - y, x + y)$ and let $G(x, y) = (x \cos y, x \sin y)$. Find the derivative of $F \circ G$ using the chain rule.
- (3) Suppose that a rotating circle of radius 1 is travelling through the plane, so that at time t seconds the center of the circle is at the point $(t, \sin t)$. Let P be the point on the circle which is at $(0, 1)$ at time $t = 0$. If the circle makes 3 revolutions per second, what is the path $\mathbf{x}(t)$ taken by the point P ?
- (4) A rotating circle of radius 1 follows a helical path in \mathbb{R}^3 so that at time t the center of the circle is at $(\sin t, \cos t, t)$. At each time t , the circle lies in the osculating plane. (That is, the circle lies in the plane spanned by the unit tangent and the unit normal vectors.) Let P be the point on the circle which is at $(1, 0)$ at time $t = 0$. The circle completes one rotation every 2π seconds. Find a formula $\mathbf{x}(t)$ for the path taken by the point P . (Hint: Express the center of the circle as a combination of the unit tangent and normal vectors.)
- (5) Explain what it means for curvature to be an intrinsic quantity.
- (6) Prove that the curvature at any point of a circle of radius r is $1/r$.
- (7) Let $\mathbf{x}(t) = (\cos t, \sin t, t)$ for $1 \leq t \leq 2$. Find \mathbf{T} , \mathbf{N} , and \mathbf{B} (that is, the moving frame) for \mathbf{x} and also find κ (the curvature).
- (8) Suppose that $\mathbf{x}: [a, b] \rightarrow \mathbb{R}^n$ is a C^1 path such that for all t , $\|\mathbf{x}(t)\| = 5$. Prove that at each t , $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ are perpendicular.
- (9) A particle is following the path $\mathbf{x}(t) = (t, t^2, t^3)$ for $1 \leq t \leq 5$. Find an integral representing the distance travelled by the particle after t seconds.
- (10) Let $\mathbf{x}(t) = (t^2, 3t^2)$ for $t \geq 1$. Reparameterize \mathbf{x} by arc length.
- (11) Suppose that $\mathbf{x}(t)$ is a path in \mathbb{R}^n such that $\mathbf{x}(0) = \mathbf{a}$ and $\mathbf{x}(1) = \mathbf{b}$ (that is, \mathbf{x} is a path joining \mathbf{a} to \mathbf{b} .) Find a path which has the same image as \mathbf{x} but which joins \mathbf{b} to \mathbf{a} .

- (12) Let $\mathbf{x}: [a, b] \rightarrow \mathbb{R}^n$ be a path with $\mathbf{x}'(t) \neq \mathbf{0}$ for all t . Let $\mathbf{y} = \mathbf{x} \circ \phi$ be an orientation reversing reparameterization of \mathbf{x} . Suppose that $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is integrable. Prove that $\int_{\mathbf{y}} f \, ds = \int_{\mathbf{x}} f \, ds$.
- (13) Let $\mathbf{x}(t) = (t \cos t, t \sin t)$ for $0 \leq t \leq 2\pi$. Let $f(x, y) = y \cos x$. Find a one-variable integral representing $\int_{\mathbf{x}} f \, ds$.
- (14) The gravitation vector field in \mathbb{R}^3 is $F(\mathbf{x}) = -\mathbf{x}/\|\mathbf{x}\|^3$. Find an integral representing the amount of work done by gravity as an object moves through the vector field F along the path $\mathbf{x}(t) = (t \cos t, t \sin t, t)$ for $1 \leq t \leq 2\pi$.
- (15) Let $\mathbf{F}(x, y) = (x, -2y)$.
- Sketch a portion of the vector field F .
 - Sketch a flow line for the vector field starting at $(1, 1)$.
 - Find a parameterization for the flow line starting at $(1, 1)$.
 - The vector field F is a gradient field. Find the potential function.
- (16) Let $F(x, y) = (2xy, x^2 + 1)$. Find a potential function for F .
- (17) Explain why flow lines for an everywhere non-zero gradient field never close up. Use this to prove that $\mathbf{F}(x, y) = (-y, x)$ is not a gradient field.
- (18) Let $f(x, y) = ye^x$. Find the gradient of f .
- (19) Suppose that $\mathbf{F} = \nabla f$ is the gradient field of a C^1 scalar field f on \mathbb{R}^2 . Let $\mathbf{x}: [a, b] \rightarrow \mathbb{R}^2$ be a C^1 path. State and prove the Fundamental Theorem of Calculus for conservative vector fields.