MA 262 Homework 8: Diverging from Normal

These first problems all involve the work done by an electric force on a moving particle. They are intended to emphasize the importance of conservative vector fields and to show another example of how line integrals are important.

Problem 8.A: Suppose that P is a particle with charge $+1$ at $\mathbf{x} \in \mathbb{R}^2$ and that Q is a particle with charge $+1$ at position $\mathbf{a} \in \mathbb{R}^2$. Coulomb's law states that the force (electric field) exerted by P on Q is (up to a constant of proportionality):

$$\mathbf{F}(\mathbf{a}) = \frac{1}{||\mathbf{a}-\mathbf{x}||^3}(\mathbf{a}-\mathbf{x})$$

(1) Show that $f(\mathbf{a}) = \frac{-1}{||\mathbf{x}-\mathbf{a}||}$ is a potential function for $\mathbf{F}$.
(2) Suppose that P is at $\mathbf{x} = \mathbf{0}$ and that Q travels along the path $\mathbf{a}(t) = \begin{pmatrix} r \\ t \end{pmatrix}$ for some fixed $r > 0$ and $0 \leq t \leq 5$. Compute the work done by the electric field exerted by P on Q. (Remember that the electric field is conservative!)

Problem 8.B: Suppose that P_1, P_2, and P_3 are particles each with charge $+1$ at positions $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3} \in \mathbb{R}^2$ and that Q is a particle with charge $+1$ at position $\mathbf{a} \in \mathbb{R}^2$. The law of superposition states that the force (electric field) exerted by P_1, P_2, and P_3 on Q is (up to a constant of proportionality):

$$\mathbf{F}(\mathbf{a}) = \sum_{i=1}^{3} \frac{1}{||\mathbf{a}-\mathbf{x_i}||^3}(\mathbf{a}-\mathbf{x_i})$$

(1) Find a potential function for $\mathbf{F}$.
(2) Suppose that $\mathbf{x_1} = \mathbf{0}$, $\mathbf{x_2} = (0,1)$ and $\mathbf{x_3} = (0,2)$. Suppose that Q travels along the path $\mathbf{a}(t) = \begin{pmatrix} r \\ t \end{pmatrix}$ for some fixed $r > 0$ and $0 \leq t \leq 5$. Compute the work done by the electric field exerted by P_1, P_2, and P_3 on Q. (Remember that the electric field is conservative!)

Problem 8.C: Suppose that $\mathscr{P}$ is a wire, each point of which has charge $+1$ and suppose that Q is a particle with charge $+1$ at position $\mathbf{a} \in \mathbb{R}^2$. Let $\mathbf{F}$ be the electric field generated by $\mathscr{P}$. Think of $\mathbf{F}$ as the force exerted by $\mathscr{P}$ on Q.

(1) Use superposition to find a potential function for $\mathbf{F}$.
(2) Suppose that $\mathscr{P}$ is the portion of the y-axis between $\mathbf{0}$ and $(0,5)$. Suppose that Q travels along the path $\mathbf{a}(t) = \begin{pmatrix} r \\ t \end{pmatrix}$ for some fixed $r > 0$ and $0 \leq t \leq 5$. Compute the work done by the electric field exerted by $\mathscr{P}$ on Q. (Remember that the electric field is conservative!) Your answer may have integrals in it.

The next few problems give you some practice with the planar divergence theorem.

Problem 8.D: Let C be the rectangle $[0,2] \times [0,1]$ and let $\mathbf{F}(x,y) = \begin{pmatrix} x^2 \\ -y^2 \end{pmatrix}$. Calculate both integrals appearing in the statement of the planar divergence theorem and verify that they are equal.

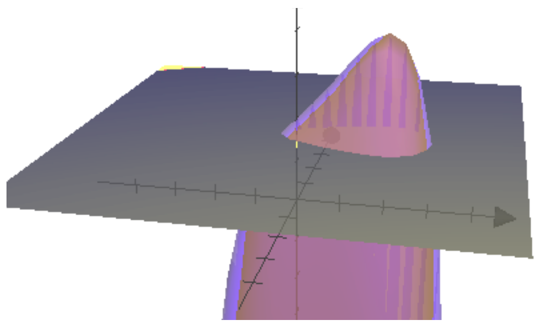
Problem 8.E: Suppose that $\mathbf{F}$ is a C^1 vector field with $\operatorname{div} \mathbf{F} = 2$ and that the flux of $\mathbf{F}$ through the unit circle (oriented counterclockwise, centered at the origin) is 10. Let C be a circle of radius 5, oriented counter-clockwise and centered at the origin. Find the flux of $\mathbf{F}$ through C.

Problem 8.F: Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a harmonic function. (This means that $\nabla^2 f = \operatorname{div} \nabla f = 0$.)

(1) Suppose that C is a closed equipotential line for f. (That is for all $(x,y) \in C$, $f(x,y)$ is constant.) Recall that $\mathbf{F} = \nabla f$ is always perpendicular to C. Use this fact and the planar divergence theorem to show that:
$$\int_C ||\mathbf{F}||\, ds = 0.$$
(2) Explain why the previous part implies that if a harmonic function has a closed equipotential line then it is constant on the region bounded by the equipotential line.

(3) Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a harmonic function. Explain why there is no point $\mathbf{a} \in \mathbb{R}^2$ such that $\mathbf{a}$ is an isolated local maximum. (A point $\mathbf{a}$ is an isolated local maximum if, for all $\mathbf{x}$ close enough to $\mathbf{a}$ but not equal to $\mathbf{a}$, $f(\mathbf{x}) < f(\mathbf{a})$.)

Hint: Suppose that $\mathbf{a}$ was an isolated local maximum and use the picture below to argue that f must have a closed contour line enclosing $\mathbf{a}$. The picture below depicts the graph of a function with an isolated local maximum and a plane of constant height cutting through it.



A word of motivation is probably in order here. As you've occasionally seen, harmonic functions are important in physics and applied mathematics. For example, the potential function for the gravitational field is a harmonic function and Socha used harmonic functions in her article on modelling water waves to find a mathematical description of certain waves. Harmonic functions are used to describe so-called "minimal surfaces", such as soap bubbles. You also know from previous homework assignments that harmonic functions are special. For example, their values on the boundary of a region determine the values inside the region. The problem you just completed gives another important property of harmonic functions: they don't have isolated maxima. A variation on the argument you just gave shows in fact that if f is a non-constant harmonic functions on a closed and bounded region then the maximum of f (which must exist because f is continuous) occurs on the boundary of the region.

Read section 7.1 and do problems 1, 5, 10 on page 417-418.

In this bonus problem, you will prove the planar divergence theorem.

Problem 8.G: (Bonus!) Suppose that $S \subset \mathbb{R}^2$ and $\mathbf{F} = \begin{pmatrix} M \\ N \end{pmatrix}$ is a vector field satisfying the hypotheses of Green's theorem.

(1) Let $\mathbf{G} = \begin{pmatrix} N \\ -M \end{pmatrix}$. Explain why $\mathbf{G}$ is just $\mathbf{F}$ rotated counterclockwise by $\pi/2$ radians. Also show that $\operatorname{scurl}\mathbf{G} = \operatorname{div}\mathbf{F}$.

(2) Let $\mathbf{n}$ be an outward unit normal to ∂C. Draw a picture to show how the effect of rotating $\mathbf{n}$ counterclockwise by $\pi/2$ is equal to $\mathbf{T}$, the unit tangent vector to ∂C.

(3) At a fixed point on C, explain why $\mathbf{G} \cdot \mathbf{T} = \mathbf{F} \cdot \mathbf{n}$ and hence why

$$\int_C \mathbf{G} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot \mathbf{n}\, ds.$$

(4) Let $C(t)$ be a parameterization of C. Use the fact that $\mathbf{T} = \frac{C'(t)}{||C'(t)||}$ to show that

$$\int_C \mathbf{G} \cdot \mathbf{T}\, ds = \int_C \mathbf{G} \cdot d\mathbf{s}.$$

(5) Apply Green's theorem to $\mathbf{G}$ and use the result to prove the planar divergence theorem for $\mathbf{F}$.