MA 262 Homework 7: Stoked about Green's Theorem

Read Sections 6.2 and 6.3. As you read pay attention to both how Green's theorem helps us do calculations and to how it helps us develop a better theoretical understanding of vector fields.

Do these problems:

pg 388: 1, 3, 7, 8, 10, 19, 24. (Problem 24 is a problem about *harmonic* functions – functions whose LaPlacian is zero. Harmonic functions play an important role in physics (eg. the study of gravity) and have many special properties. This is one of them.)

pg 399: Pick two problems of your choice from 3-13 and use curl to determine if the field is conservative. You don't have to find the potential function. Remember that you can only use curl to determine if the region under consideration is simply connected.

pg 403: Read the italicized text and also read problems 30 - 33. Do problem 32. For that problem, you need to assume the function is harmonic. The point of these problems is that harmonic functions have the very special property that their values inside a region are determined by their values on the boundary. This is like saying that the value of the function where you are is determined by the values of the function at points 1000 miles from where you are – very strange! The problem is best done by being clear on the definition of normal derivative and recognizing that $\nabla f \cdot \nabla f = ||\nabla f||^2$. Combine this with the definition of $\partial f/\partial n$ (given in problem 30) and with exercise 31, to get your answer.

Problem 7.A Let $\mathbf{F} = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$. Recall that you studied this vector field in HW 6.

(1) In HW 6, you showed that $\operatorname{scurl}\mathbf{F} = 0$ and that $\int_C \mathbf{F} \cdot d\mathbf{s} = 2\pi$ where C is the unit circle, centered at the origin and oriented counter-clockwise. Explain why these facts don't contradict Green's theorem.

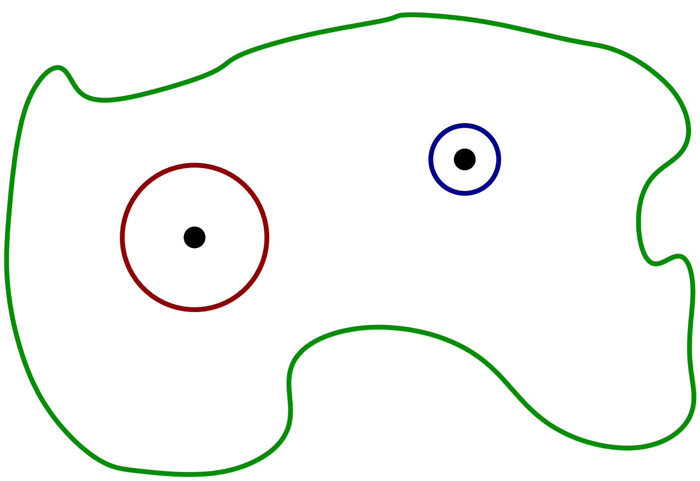
(2) Let $\mathbf{a} \in \mathbb{R}^2$ and let r be a number such that $0 < r < ||\mathbf{a}||$. Let P_r be the circle centered at $\mathbf{a}$ and of radius r, oriented counter-clockwise.

It can be paramterized by $P_r(t) = (r\cos t, r\sin t) + \mathbf{a}$ for $0 \leq t \leq 2\pi$. Use Green's theorem to calculate $\int_{P_r} \mathbf{F} \cdot d\mathbf{s}$. Be sure to explain why you can apply Green's theorem to the region enclosed by P_r.

(3) Let Q_r be the circle of radius r centered at the origin and oriented counterclockwise. Assuming $r \neq 1$, use Green's theorem to show that
$$\int_{Q_r} \mathbf{F} \cdot d\mathbf{s} = \int_C \mathbf{F} \cdot d\mathbf{s} = 2\pi,$$
where C is the unit circle centered at the origin. (Hint: Consider the region between Q_r and C.)

(4) Let ϕ be any closed C^1 curve in $\mathbb{R}^2$ oriented counterclockwise. ("Closed" means it joins up to itself to make a loop.) Prove that if ϕ encloses the origin then $\int_\phi \mathbf{F} \cdot d\mathbf{s} = 2\pi$ and if ϕ does not enclose the origin then $\int_\phi \mathbf{F} \cdot d\mathbf{s} = 0$. (Hint: Use the same idea as in part (3), but look at a region between a small circle and ϕ.)

Problem 7.B Suppose that $\mathbf{F}$ is a C^1 vector field defined on all points of $\mathbb{R}^2$ except two special points $\mathbf{p}_1$ and $\mathbf{p}_2$. Assume that $\operatorname{scurl} \mathbf{F} = 0$. Let C_1 be a circle enclosing $\mathbf{p}_1$ but not $\mathbf{p}_2$ and let C_2 be a circle enclosing $\mathbf{p}_2$ but not $\mathbf{p}_1$. Let C_3 be a closed C^1 curve enclosing both points. (See the picture below, where C_1 is red; C_2 is blue; and C_3 is green.)



Assume that $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = 29$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{s} = 200\pi$. Find $\int_{C_3} \mathbf{F} \cdot d\mathbf{s}$.